Lecture 3: Linear Systems - Part 2

Winter 2016

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Sketching behaviors of linear models, and the zoo of all possible solutions for second order linear ODE systems.

Leonhard Euler
1707 - 1783

Joseph Fourier
1768 - 1830

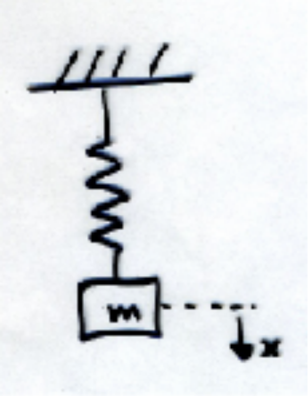
Henri Poincare
1854 - 1912

So, today we do two things: we learn to **think qualitatively** about behaviors of linear dynamical systems and show that these systems are **globally understandable**.

	n = 1	n = 2 or 3	n >> 1	continuum
Linear	exponential growth and decay single step conformational change fluorescence emission pseudo first order kinetics	second order reaction kinetics linear harmonic oscillators simple feedback control sequences of conformational change	electrical circuits molecular dynamics systems of coupled harmonic oscillators equilibrium thermodynamics diffraction, Fourier transforms	Diffusion Wave propagation quantum mechanics viscoelastic systems
Nonlinear	fixed points bifurcations, multi stability irreversible hysteresis overdamped oscillators	anharmomic oscillators relaxation oscillations predator-prey models van der Pol systems Chaotic systems	systems of non-linear oscillators non-equilibrium thermodynamics protein structure/ function neural networks the cell ecosystems	Nonlinear wave propagation Reaction-diffusion in dissipative systems Turbulent/chaotic flows

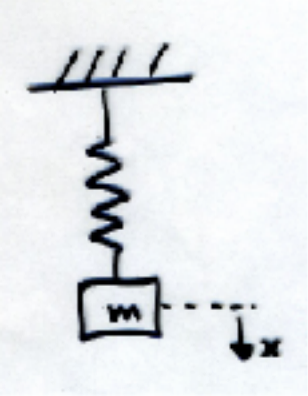
adapted from S. Strogatz

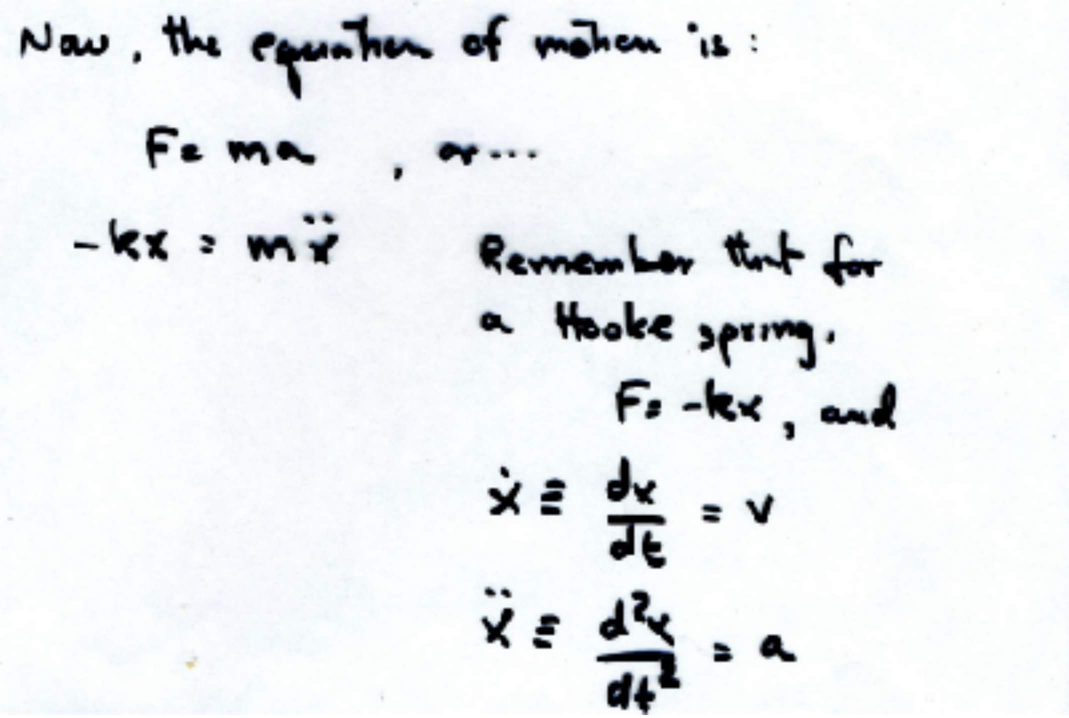
Qualitative analysis of system dynamics…



Often, for many complex systems, it is hard to get analytic solutions or to be intuitive about their behavior. We need a way of “seeing” system behavior. To do this, let us begin with the simple harmonic oscillator...

Qualitative analysis of system dynamics…




Now, the equation of motion is:

$F = ma$, or...

$$-kx = m\ddot{x}$$

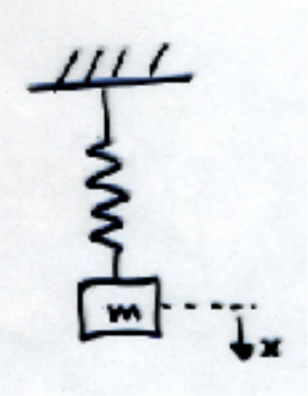
Remember that for a Hooke spring,

$F = -kx$, and

$$\dot{x} \equiv \frac{dx}{dt} = v$$

$$\ddot{x} \equiv \frac{d^2x}{dt^2} = a$$

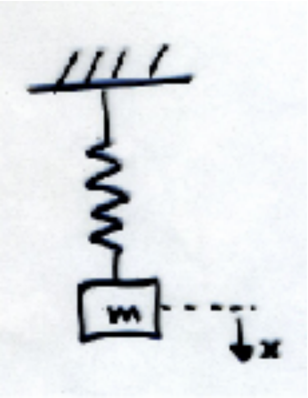
Qualitative analysis of system dynamics…



So... $\boxed{m\ddot{x} + kx = 0}$

We can re-write this equation. We note that the system is fully characterized its position and velocity...

Qualitative analysis of system dynamics…



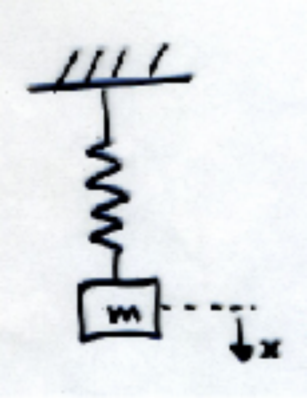
So… $\boxed{m\ddot{x} + kx = 0}$

$$\dot{x} = v$$

$$\dot{v} = -\frac{k}{m}x$$

The first equation is just the definition of velocity. The second equation is $m\ddot{x} + kx = 0$ rewritten in terms of v.

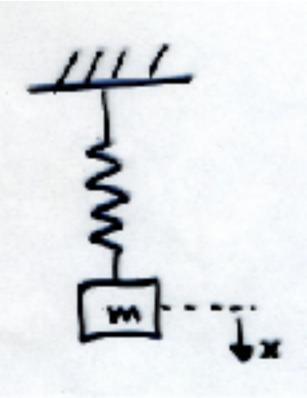
Qualitative analysis of system dynamics…



To simplify, we define $\omega^2 = \frac{k}{m}$. So..

$$\boxed{\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\omega^2 x\end{aligned}}$$

Qualitative analysis of system dynamics…

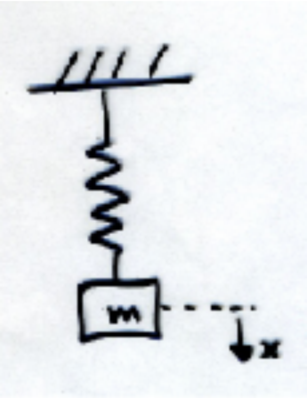


$$\boxed{\begin{aligned}\dot{x} &= v\\ \dot{v} &= -\omega^2 x\end{aligned}}$$

Thus, for every (x,v), this system of equations assigns a vector $(\dot{x},\dot{v}) = (v, -\omega^2 x)$. This makes a vector field…

....for this system, (x,v) represents a 2D “phase space” in which we can see the behavior of the system intuitively.

Qualitative analysis of system dynamics…



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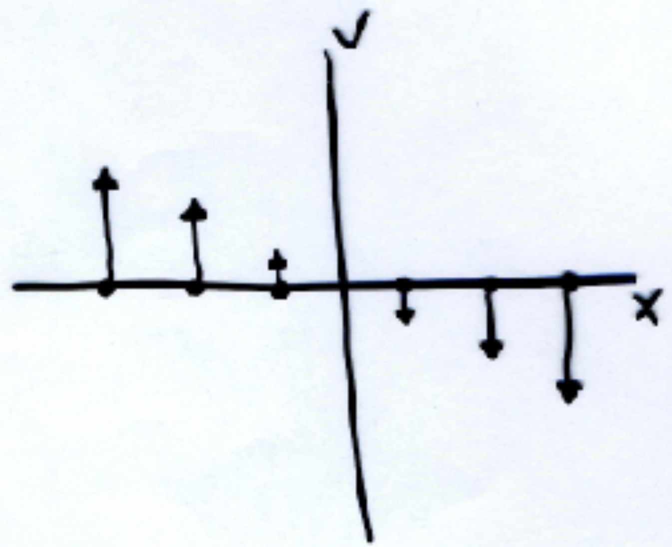


① If on the x-axis (v = 0)

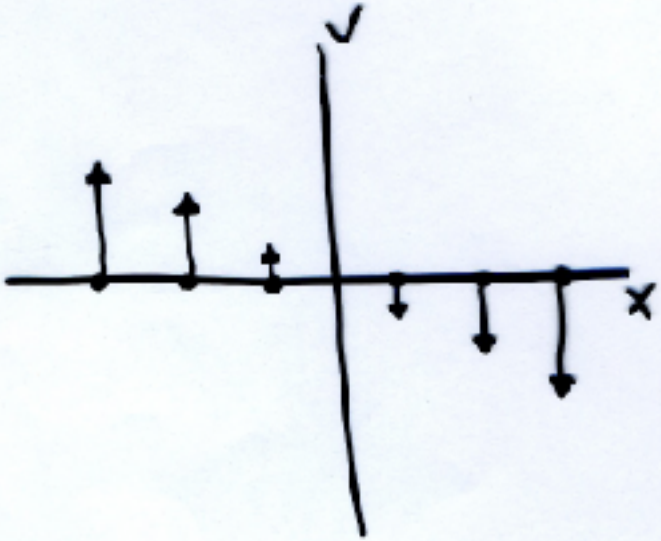
$(\dot{x},\dot{v}) = (0, -\omega^2 x)$

⇒ negative for +x

⇒ positive for −x

Qualitative analysis of system dynamics…

$$\boxed{\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\omega^2 x\end{aligned}}$$



① If on the x-axis (v = 0)

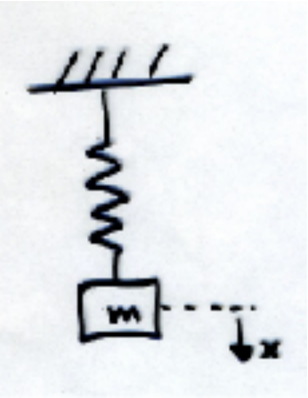
$(\dot{x},\dot{v}) = (0,-\omega^2 x)$

⇒ negative for +x

⇒ positive for −x

The places where the derivatives of our system variables go to zero are called the “**nullclines**”…

Qualitative analysis of system dynamics…



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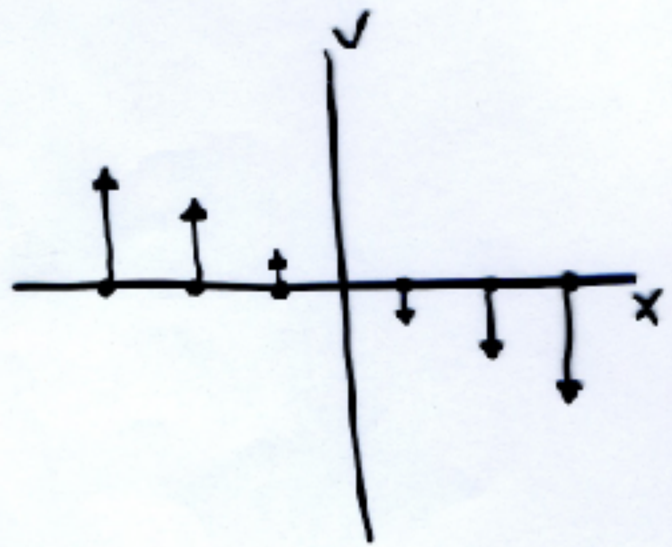


① If on the x-axis (v=0)

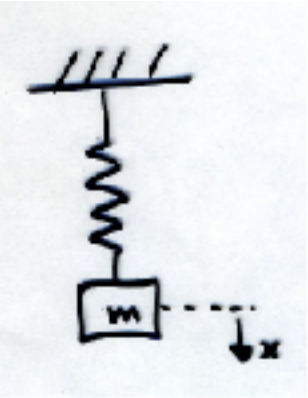
$(\dot{x},\dot{v}) = (0, -\omega^2 x)$

⇒ negative for +x

⇒ positive for −x

…and the other **nullcline**?

Qualitative analysis of system dynamics…



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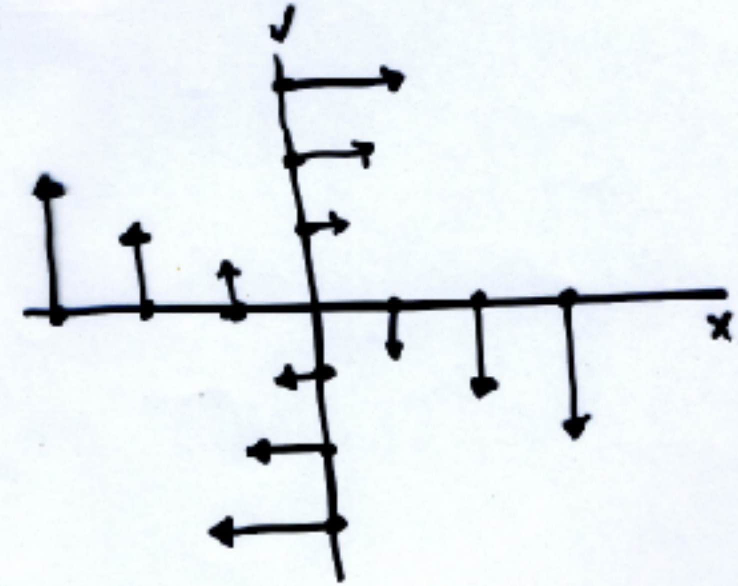


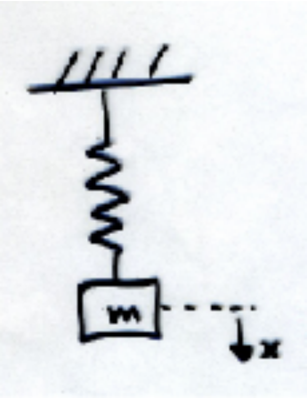
① on y-axis ($x=0$), so...

$(\dot{x}, \dot{v}) = (v, 0)$

=> positive for $+v$

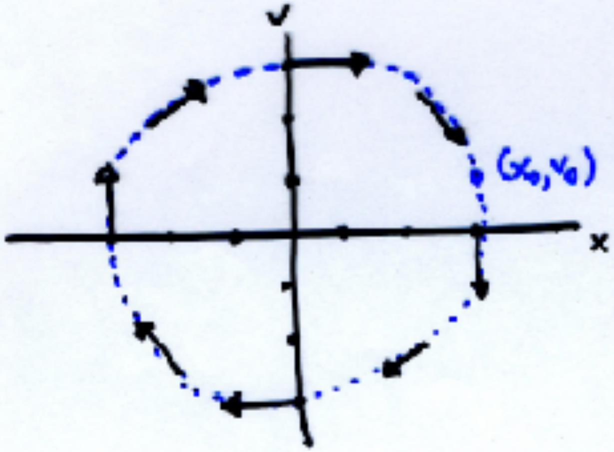
=> negative for $-v$

Qualitative analysis of system dynamics…



$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x$$

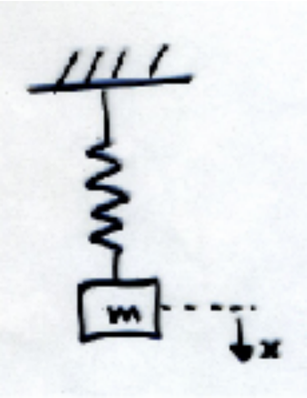
Now, imagine the vector field as a sort of imaginary "fluid" that carries you around the phase space...



So... you flow around the origin.

At the origin, all flows are zero, so you stay put... a fixed point

Qualitative analysis of system dynamics…



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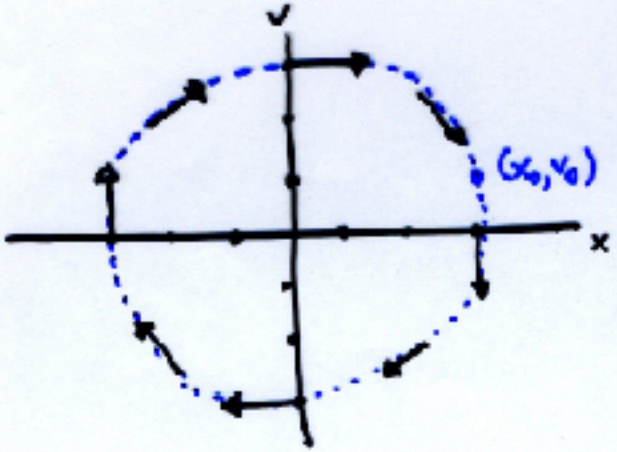

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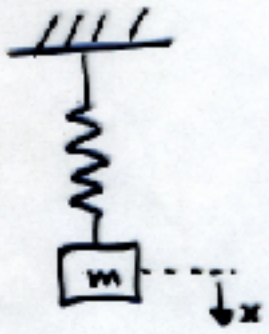
At the origin, all flows are zero, so you stay put... a fixed point

This is called a "**phase portrait**"…a way of seeing system dynamics.

Qualitative analysis of system dynamics…



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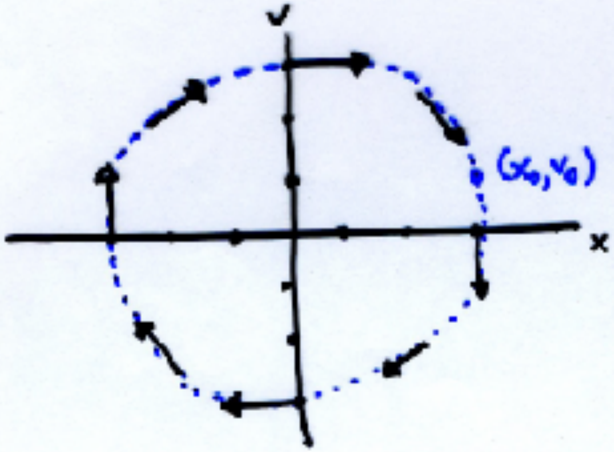
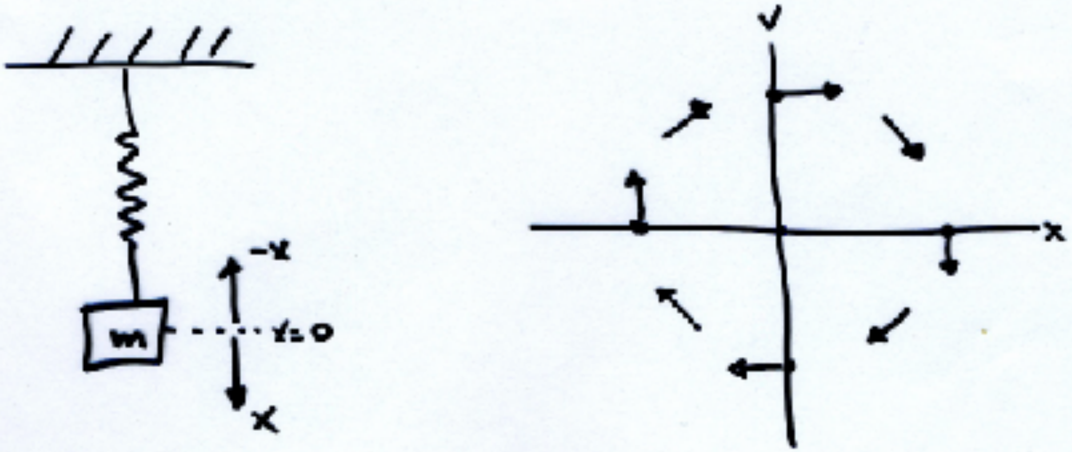


So… you flow around the origin.

At the origin, all flows are zero, so you stay put… a fixed point

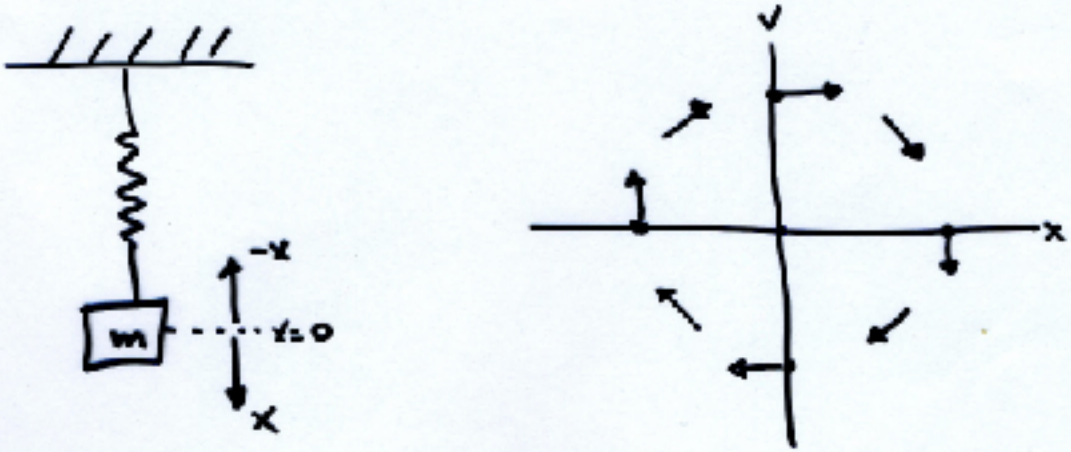
Here, the flow comes back around to the starting point (x_0, v_0) and so the trajectory is called a closed orbit

Qualitative analysis of system dynamics…



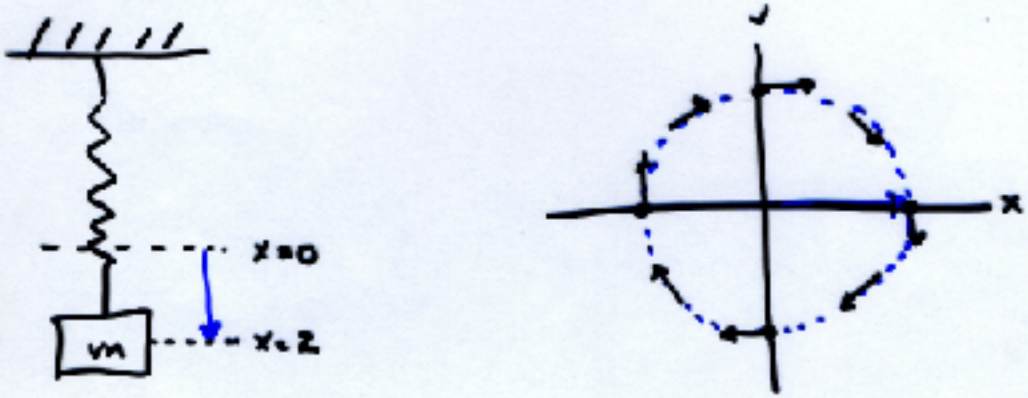
So… ① at rest, the mass is at $(x,v) = (0,0)$ and it stays right there forever

Qualitative analysis of system dynamics…



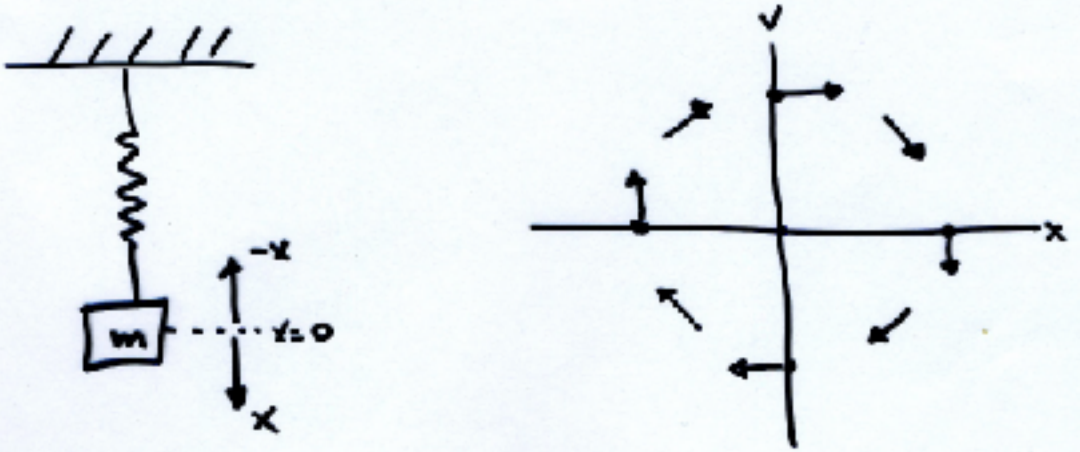
So... ① at rest, the mass is at (x,v)= (0,0) and it stays

② If displaced to positive x ...



... it oscillates, and goes on forever.

Qualitative analysis of system dynamics…



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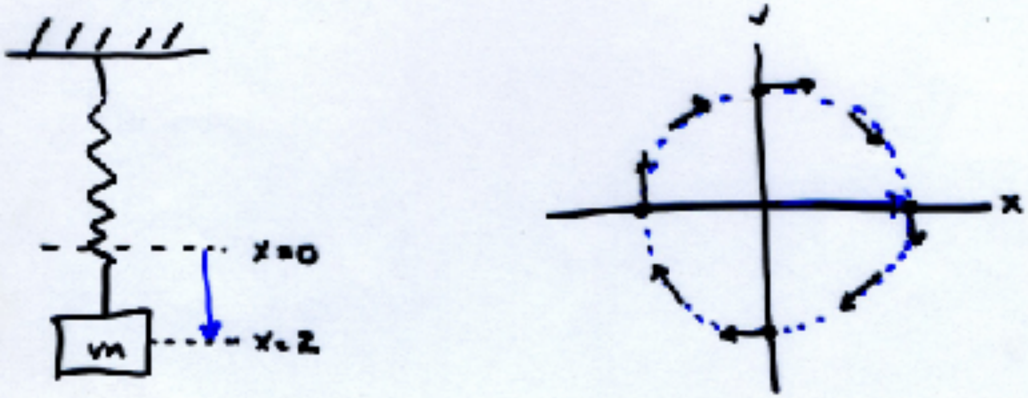


Quite reasonably,

maximum extension at zero velocity, and…

maximum velocity at zero displacement..

... it oscillates, and goes on forever.

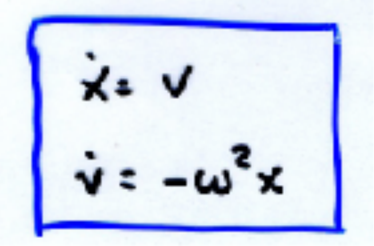
Qualitative analysis of system dynamics…

One more example…..here is another system of two equations.

$$\dot{x} = ax$$

$$\dot{y} = -y$$

How is it different from our previous case?



$$\boxed{\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\omega^2 x\end{aligned}}$$

adapted from S. Strogatz

Qualitative analysis of system dynamics…

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$$\dot{x} = ax$$
$$\dot{y} = -y$$

How is it different from our previous case?

$$\boxed{\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\omega^2 x\end{aligned}}$$

The top equations are said to be “**uncoupled**”… and therefore easy to solve right?

adapted from S. Strogatz

Qualitative analysis of system dynamics…

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y \end{aligned} \quad \longrightarrow \quad \begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t} . \end{aligned}$$

given the obvious initial conditions…

We can sketch the flow in the x - y plane for various values of **a**…

adapted from S. Strogatz

Qualitative analysis of system dynamics…

$$\begin{aligned}\dot{x} &= ax \\ \dot{y} &= -y\end{aligned} \quad \longrightarrow \quad \begin{aligned}x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t}.\end{aligned}$$

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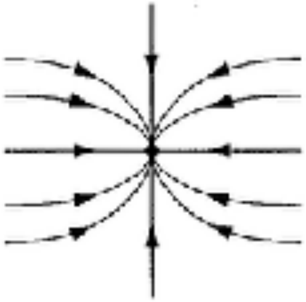


(a) $a < -1$

For **a < -1**:

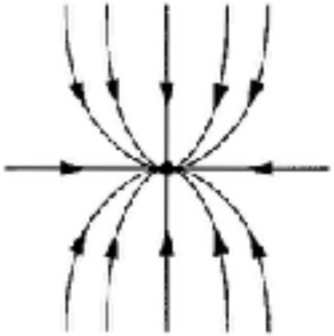
(1) The flow in both directions eventually goes to the origin as expected. The origin is the “fixed point” of the system….the only place where there is no flow.
(2) The origin is stable…slight perturbations from it will make the system relax back to the origin. The origin is called a **stable node** or **stable fixed point**.
(3) The flow is faster in the x-direction, and so all the flow lines arrive at the origin along the y-direction (the slower one).

adapted from S. Strogatz

Qualitative analysis of system dynamics…

$$\begin{aligned}\dot{x} &= ax \\ \dot{y} &= -y\end{aligned} \quad \longrightarrow \quad \begin{aligned}x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t}.\end{aligned}$$

We can sketch the flow in the x - y plane for various values of **a**…



(c) $-1 < a < 0$

For **<u>-1 < a < 0</u>**:

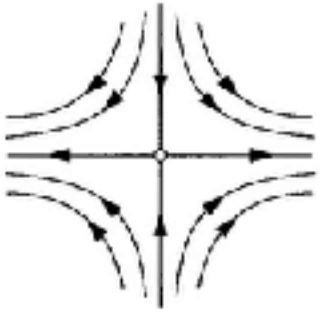
(1) The flow in both directions still asymptotically goes to the origin
(2) The origin is still a **stable fixed point**…slight perturbations from it will make the system relax back to the origin.
(3) But flow is now faster in the y-direction, and so all the flow lines arrive at the origin along the x-direction (the slower one).

adapted from S. Strogatz

Qualitative analysis of system dynamics…

$$\begin{aligned}\dot{x} &= ax \\ \dot{y} &= -y\end{aligned} \quad \longrightarrow \quad \begin{aligned}x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t}.\end{aligned}$$

We can sketch the flow in the x - y plane for various values of **a**…



(c) $a > 0$

For **<u>a > 0</u>**:

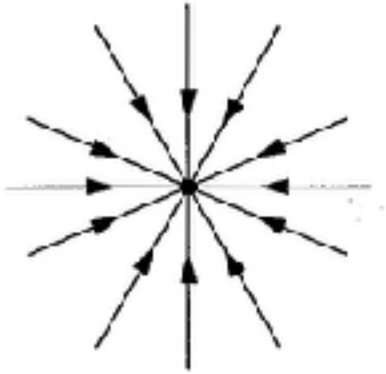
(1) The origin is still a fixed point, but the flow only goes to the origin if the initial condition starts exactly on the y-axis. Otherwise, it diverges to infinity along the x-direction.
(2) The origin is NOT stable…slight perturbations from it will make the system fly off in the x-direction.
(3) The origin is now called a **saddle point**…this happens when one of the exponentials is positive and one is negative.

adapted from S. Strogatz

Qualitative analysis of system dynamics…

$$\dot{x} = ax \qquad \dot{y} = -y \quad \longrightarrow \quad x(t) = x_0 e^{at} \qquad y(t) = y_0 e^{-t}.$$

We can sketch the flow in the x - y plane for various values of **a**…



(b) $a = -1$

For **a = -1**:

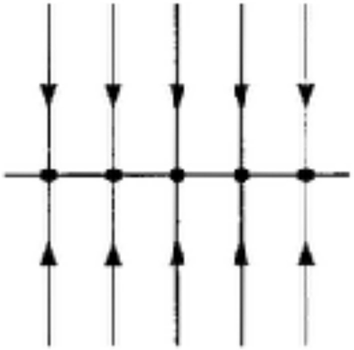
(1) The origin is a stable fixed point, and the flow is equal in both directions.
(2) The origin is now called a **symmetric node** or a **star**…

adapted from S. Strogatz

Qualitative analysis of system dynamics…

$$\begin{array}{lcl} \dot{x} = ax & & x(t) = x_0 e^{at} \\ & \longrightarrow & \\ \dot{y} = -y & & y(t) = y_0 e^{-t}. \end{array}$$

We can sketch the flow in the x - y plane for various values of **a**…



(d) $a = 0$

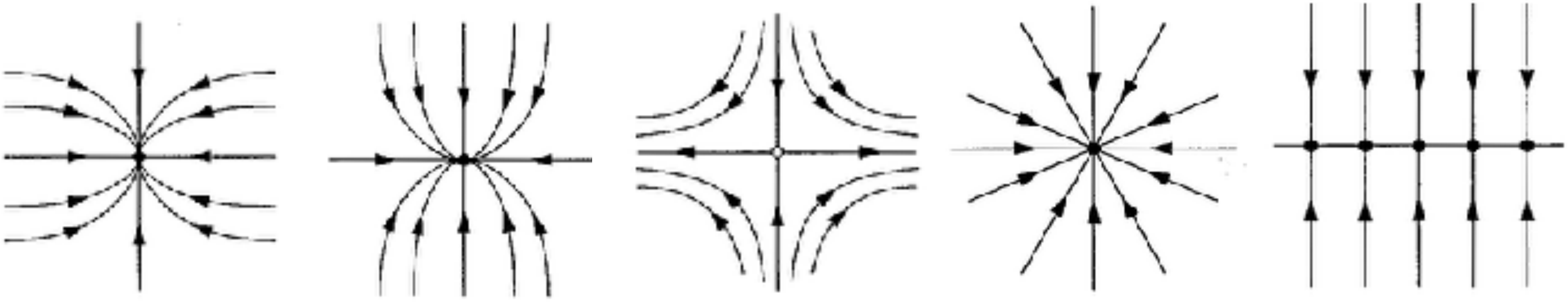
For **<u>a = 0</u>**:

(1) Now there is a whole row of fixed points….the entire x-axis is a set of stable fixed points depending on the initial conditions…a set of **degenerate nodes**.

adapted from S. Strogatz

Qualitative analysis of system dynamics…

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y \end{aligned} \quad \longrightarrow \quad \begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t}. \end{aligned}$$



(a) $a < -1$

(c) $-1 < a < 0$

(c) $a > 0$

(b) $a = -1$

(d) $a = 0$

Seems like a lot of different and maybe "**complex**" behaviors? Not really….

adapted from S. Strogatz

Let’s generalize….

$$\dot{x} = ax$$
$$\dot{y} = -y$$

We have been working with a system of **uncoupled** first-order differential equations…

Let’s generalize….

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where a, b, c, and d are constants.

What about the more general case of a system of **coupled** reactions?

First, we will introduce a **matrix notation** to write the system of equations…

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where a, b, c, and d are constants. If we use boldface to denote vectors….

↓

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where… $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

thus…

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let’s generalize….

$$\dot{x} = ax$$
$$\dot{y} = -y$$

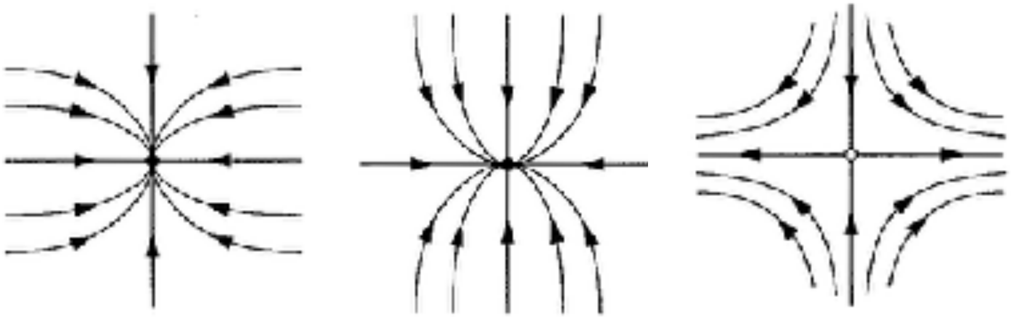
or….

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

…the uncoupled system is characterized by a **diagonalized characteristic matrix**

Let’s generalize….

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \longrightarrow \quad \begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t}. \end{aligned}$$



(a) $a<-1$ (c) $-1<a<0$ (c) $a>0$

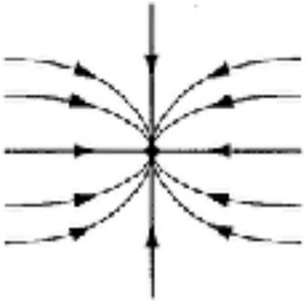
…the **x** and **y** axes are special….

(1) they represent directions of the system trajectory as t goes to +/- infinity, and…

(3) they define **straight line trajectories** along which the system will stay forever and show exponential growth or decay

Let’s generalize….

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}a & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \longrightarrow \begin{aligned}x(t) &= x_0 e^{at}\\ y(t) &= y_0 e^{-t}.\end{aligned}$$



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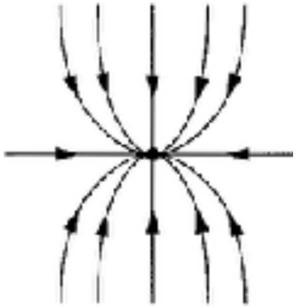

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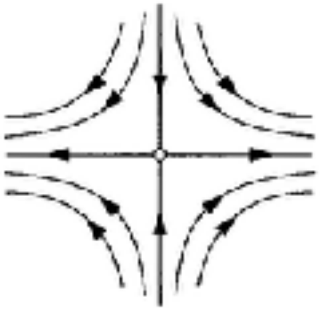


(e) $a > 0$

x(t) and y(t) are the **natural functions** for this system whose additive combination defines the behavior of the system for any value of t…

Ok, we have been considering a special case….

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}a & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

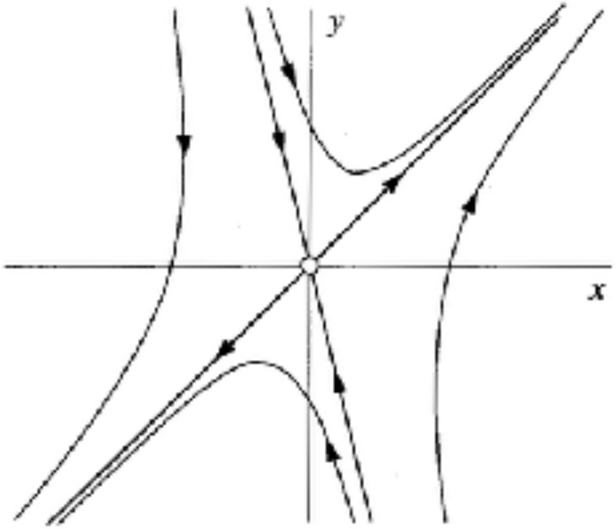
But, what about for a more **general** second-order linear system?

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

Now, what are the "**natural**" functions whose additive combination help us describe the possible behaviors of the system?

But, what about for a more **general** second-order linear system?

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Well, the **straight-line trajectories** that define the system behavior won't be as simple as just on the x and y axes (since the system is coupled). How can we find them?

The **general solution** to second-order linear system…

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\dot{\mathbf{x}} = A\mathbf{x}$ given $\mathbf{x_0}$ …a vector of initial conditions

$$\Big\downarrow$$

$$\mathbf{x}(t) = e^{\mathbf{\Lambda}t}\mathbf{x_0}$$

….where **A** is a **matrix exponential.** How do we compute it?

The **general solution** to second-order linear system…

If $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ then $e^{\mathbf{A}} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

…we would like to get **A** into a form that makes it easy to compute the matrix exponential. What should we do to it?

The **general solution** to second-order linear system…

If $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ then $e^{\mathbf{A}} = \begin{bmatrix} e^{a} & 0 \\ 0 & e^{b} \end{bmatrix}$

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

$$A = V \cdot \lambda \cdot V^T$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

…for any square, symmetric, “positive-definitive” matrix, this decomposition is always available…the so-called “**eigenvalue decomposition**”

The **general solution** to second-order linear system…

$$A = V \cdot \lambda \cdot V^T$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

↓ eigenvector 1

↓ eigenvector 2

In this process, a matrix is decomposed into its **eigenvalues** and associated **eigenvectors**….

This is quite generally important and will be covered in the mathematics course, but for now, let see how this gives us **intuitive solutions** to our general second order system…

The **general solution** to second-order linear system…

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\dot{\mathbf{x}} = A\mathbf{x}$ given $\mathbf{x_0}$ …a vector of initial conditions

$$\Big\downarrow$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x_0}$$

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$$\mathbf{x}(t) = e^{\mathbf{\Lambda} t}\mathbf{x_0}$$

$$\Big\downarrow \quad \begin{aligned} \mathbf{A} &= \mathbf{V}\lambda\mathbf{V}^\top \\ e^{\mathbf{A}} &= \mathbf{V}e^{\lambda}\mathbf{V}^\top \end{aligned}$$

The **general solution** to second-order linear system…

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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$$\Big\downarrow \quad \begin{aligned} \mathbf{A} &= \mathbf{V}\lambda\mathbf{V}^\top \\ e^{\mathbf{A}} &= \mathbf{V}e^{\lambda}\mathbf{V}^\top \end{aligned}$$

$$\mathbf{x}(t) = \mathbf{V}e^{\lambda t}\mathbf{V}^\top\mathbf{x}_0$$

So…the straight-line trajectories we are looking for are **eigenvectors of A**, and each associated **eigenvalue** gives the growth (or decay) rate along that eigenvector…

The **general solution** to second-order linear system…

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\dot{\mathbf{x}} = A\mathbf{x}$ given $\mathbf{x_0}$ …a vector of initial conditions

$$\downarrow$$

$$\mathbf{x}(t) = e^{\mathbf{\Lambda}t}\mathbf{x_0}$$

$$\downarrow$$

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v_1} + \alpha_2 e^{\lambda_2 t}\mathbf{v_2}$$

So…these so-called “**eigenfunctions**” are the natural solutions to the general case of a **linear** system….the functions whose **additive combination** defines the behavior of the system for any value of t…

The behaviors of any second order linear system of differential equations as a **linear combination** of its **eigenfunctions…**an example:

$$\begin{aligned}\dot{x} &= x + y \\ \dot{y} &= 4x - 2y\end{aligned}$$

or….

$$\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 1 \\ 4 & -2\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}$$

compute the eigenvalues and associated eigenvectors of **A**…

adapted from S. Strogatz

The behaviors of any second order linear system of differential equations as a **linear combination** of its **eigenfunctions…**an example:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

compute the eigenvalues and associated eigenvectors of **A**…

$$\lambda_1 = 2, v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3, v_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

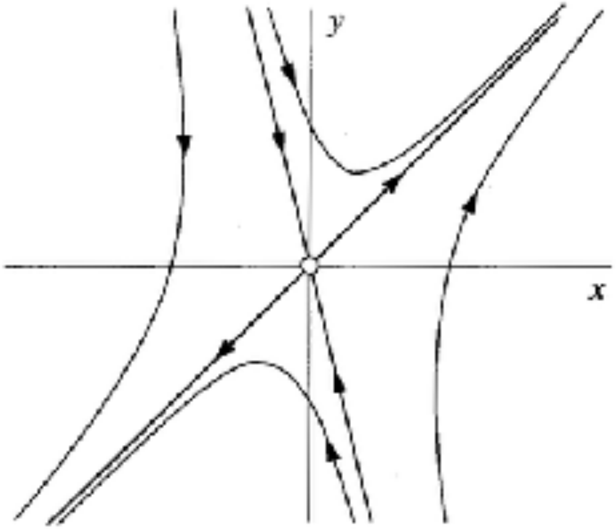
sketch the system behavior….

adapted from S. Strogatz

The behaviors of any second order linear system of differential equations as a **linear combination** of its **eigenfunctions…**an example:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 1\\ 4 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\lambda_1 = 2, v_1 = \begin{bmatrix}1\\ 1\end{bmatrix} \quad \lambda_2 = -3, v_2 = \begin{bmatrix}1\\ -4\end{bmatrix}$$



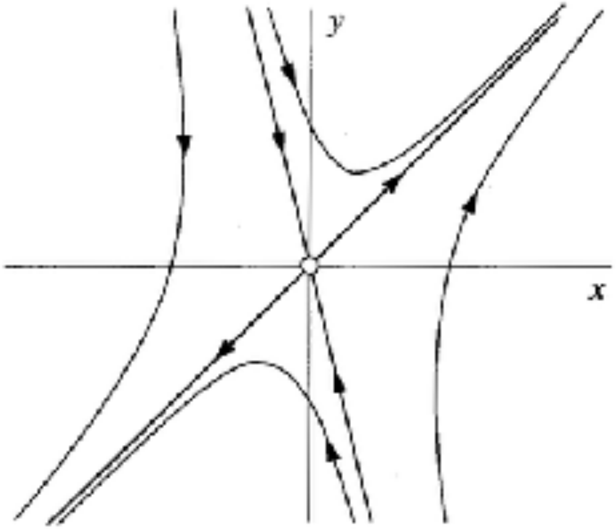
does it make sense?

adapted from S. Strogatz

The behaviors of any second order linear system of differential equations as a **linear combination** of its **eigenfunctions…**an example:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 1\\ 4 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\lambda_1 = 2, v_1 = \begin{bmatrix}1\\ 1\end{bmatrix} \quad \lambda_2 = -3, v_2 = \begin{bmatrix}1\\ -4\end{bmatrix}$$



(1) is there a fixed point? is it stable?

(2) what is the flow along eigenvector 1?

(3) what is the flow along eigenvector 2?

(4) for any initial condition, can you “see” the system behavior?

Now for the **zoo of possible solutions**….

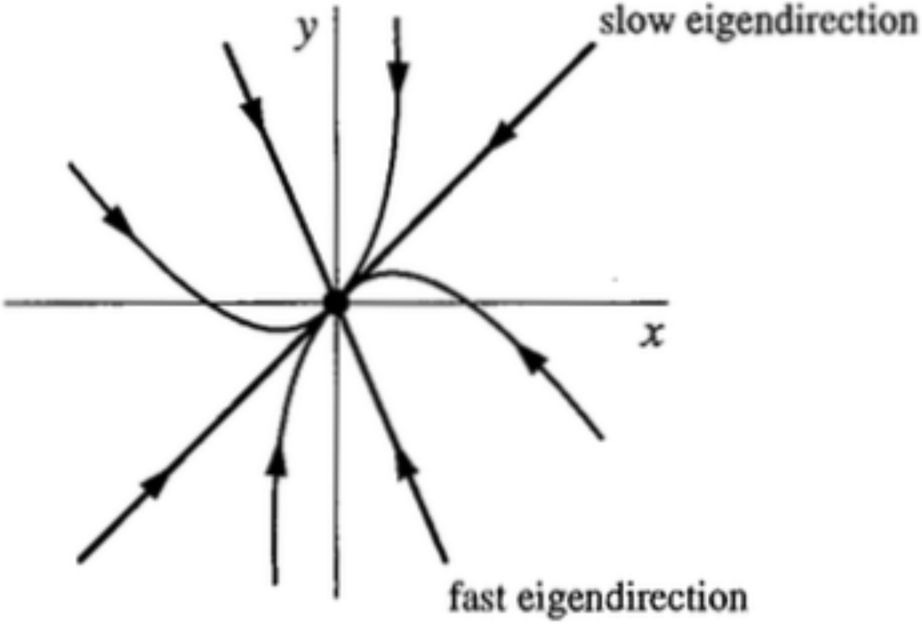
$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 t}\mathbf{v_2}$$

…note that the **eigenvalues** control the
behavior of the system

Now for the **zoo of possible solutions**….

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 t}\mathbf{v}_2$$

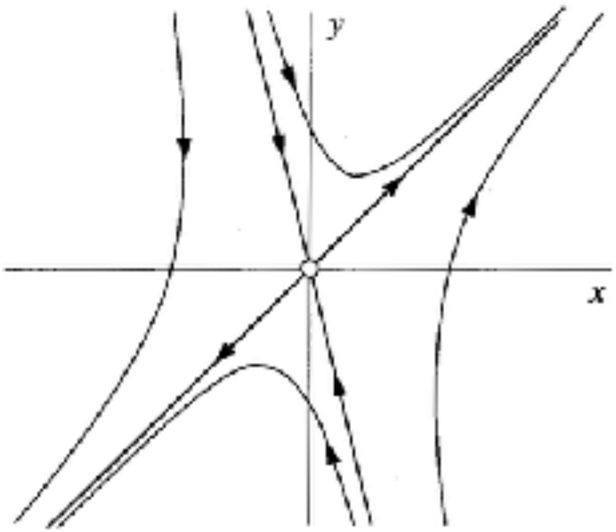
…if $\lambda_2 < \lambda_1 < 0$



Now for the **zoo of possible solutions**….

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 t}\mathbf{v_2}$$

…if $\lambda_2 > 0, \lambda_1 < 0$

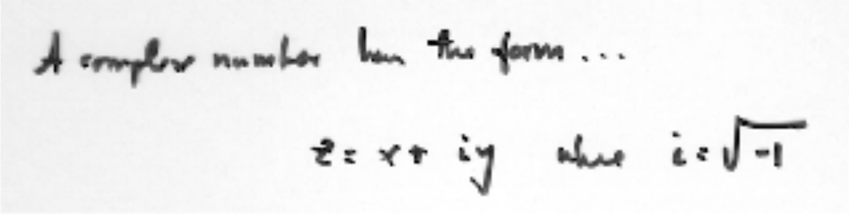


Now for the **zoo of possible solutions**….

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 t}\mathbf{v_2}$$

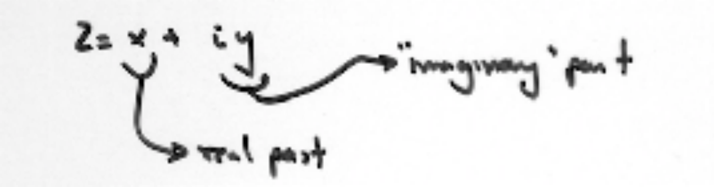
what if the eigenvalues are **complex numbers**?

Some mathematical preliminaries…complex numbers!



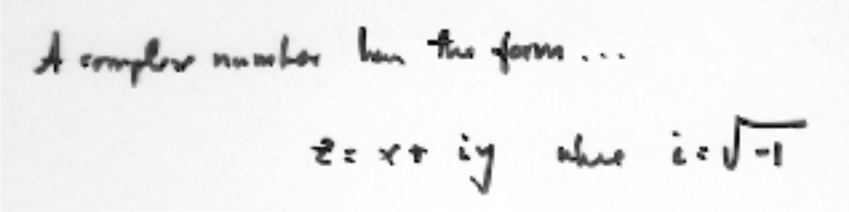
A complex number has the form . . .

$$z = x + iy \quad \text{where} \quad i = \sqrt{-1}$$



$$z = \underbrace{x}_{\text{real part}} + \underbrace{iy}_{\text{"imaginary" part}}$$

Some mathematical preliminaries…complex numbers!



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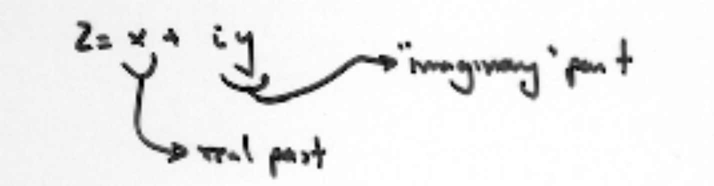
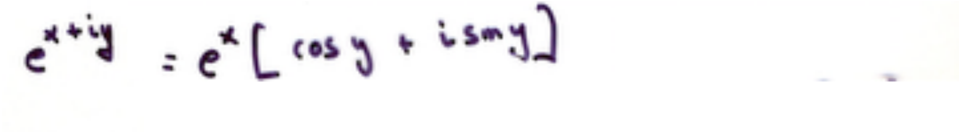


$$z = \underbrace{x}_{\text{real part}} + i\underbrace{y}_{\text{"imaginary" part}}$$



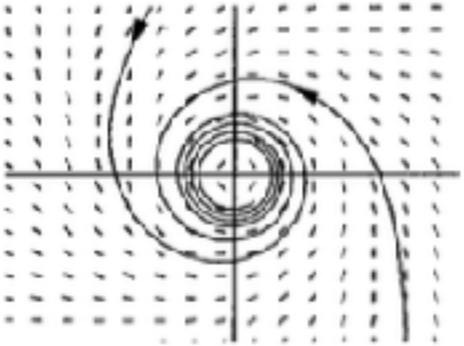
$$e^{x+iy} = e^{x}[\cos y + i\sin y]$$

Because of the **Euler relationship**…one of the great formula's of mathematics….

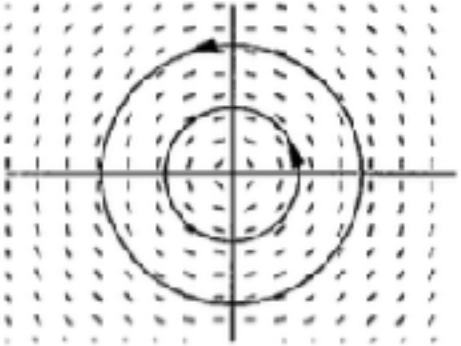
Now for the **zoo of possible solutions**….

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 t}\mathbf{v_2}$$

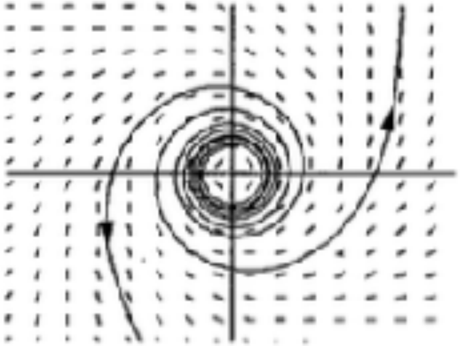
what if the eigenvalues are **complex numbers**? $\lambda_{1,2} = a \pm ib$



$a<0$



$a=0$



$a>0$

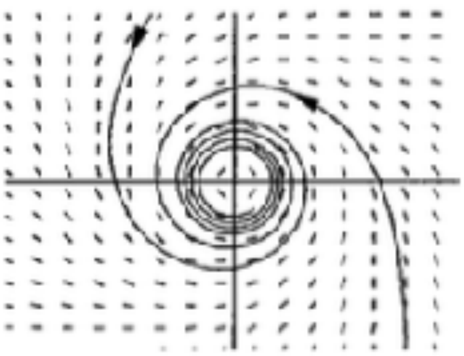
$e^{x+iy} = e^{x}[\cos y + i \sin y]$

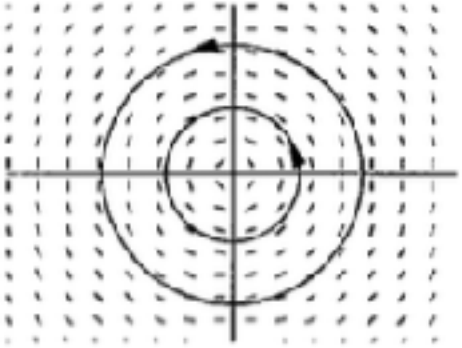
damped, constant, or growing **oscillations**….

Now for the **zoo of possible solutions**….

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v_1} + \alpha_2 e^{\lambda_2 t}\mathbf{v_2}$$

what if the eigenvalues are **complex numbers**? $\lambda_{1,2} = a \pm ib$


$a<0$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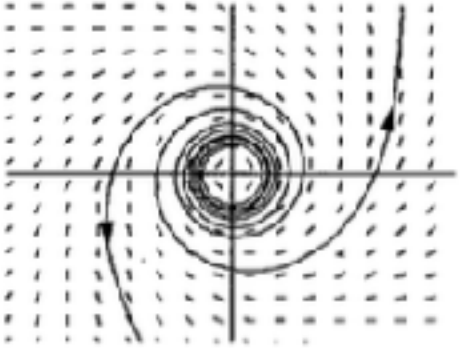
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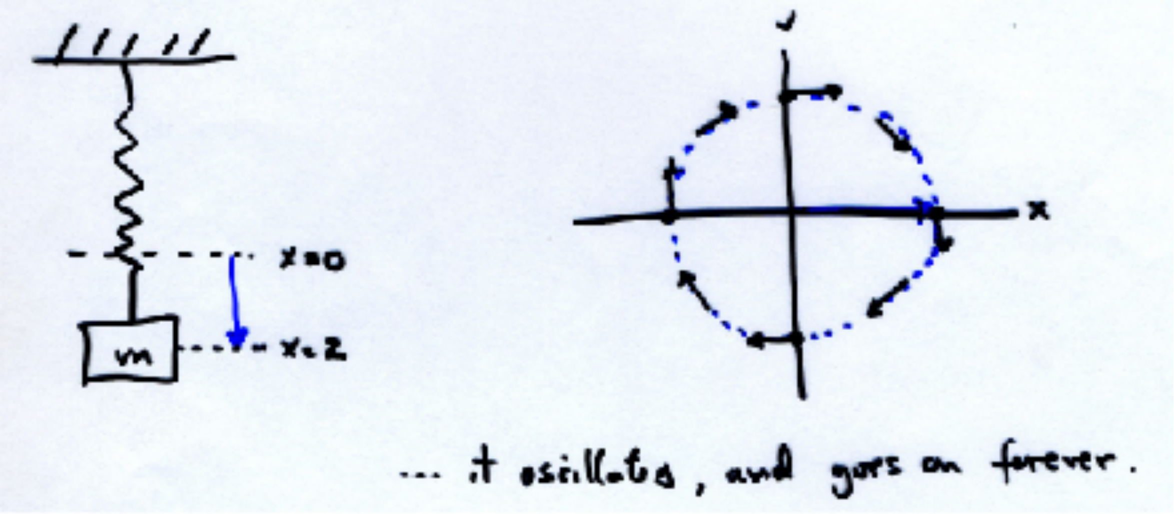
remind us of anything?

Now for the **zoo of possible solutions**….

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 t}\mathbf{v}_2$$



…same as the **linear harmonic oscillator**, one example of a linear second order system.

Now for the **zoo of possible solutions**….

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v_1} + \alpha_2 e^{\lambda_2 t}\mathbf{v_2}$$

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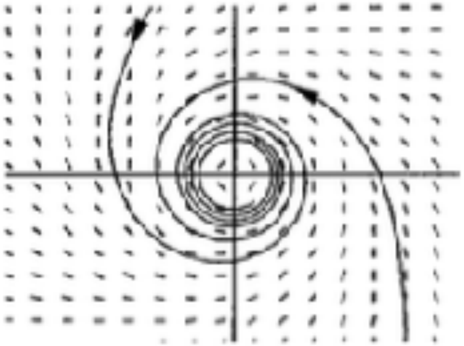
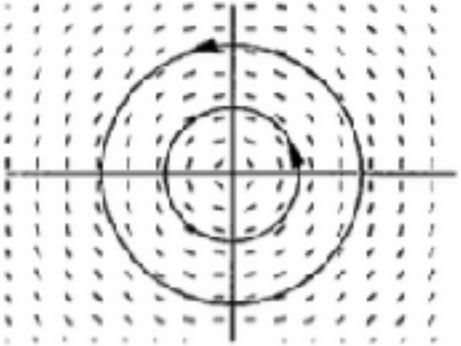


$a<0$



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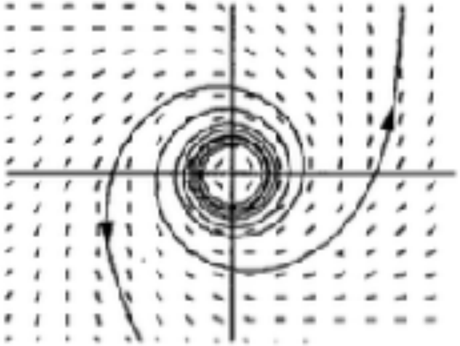


$a>0$

and how would one get this?

Now for a cool thing….

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{aligned} \tau &= \mathrm{trace}(A) = a + d\,, \\ \Delta &= \det(A) = ad - bc\,. \end{aligned}$$

…the **trace** and **determinant** of a matrix

Now for a cool thing….

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{array}{l} \tau = \text{trace}(A) = a + d\,, \\ \Delta = \det(A) = ad - bc\,. \end{array}$$

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

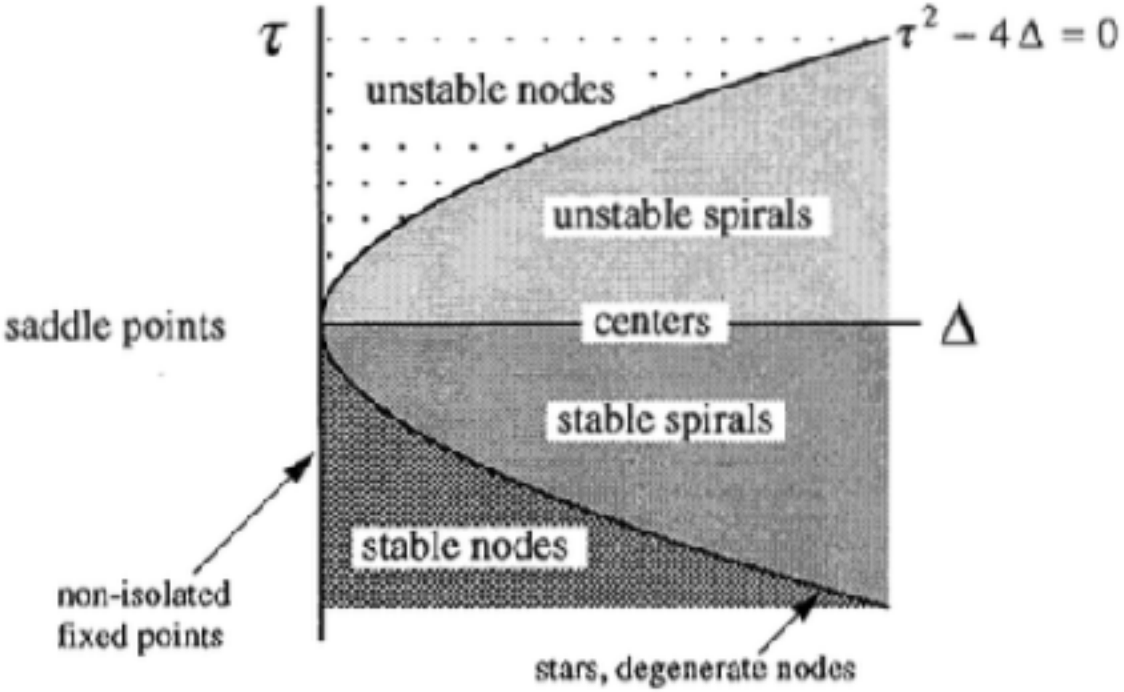
…or

$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

…the **eigenvalues** are completely determined by the **trace** and **determinant**…

Now for a cool thing….

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$



…the **zoo of all possible behaviors** for a linear, second-order system

Next time…behaviors at the stochastic limit

	n = 1	n = 2 or 3	n >> 1	continuum
Linear	exponential growth and decay single step conformational change fluorescence emission pseudo first order kinetics	second order reaction kinetics linear harmonic oscillators simple feedback control sequences of conformational change	electrical circuits molecular dynamics systems of coupled harmonic oscillators equilibrium thermodynamics diffraction, Fourier transforms	Diffusion Wave propagation quantum mechanics viscoelastic systems
Nonlinear	fixed points bifurcations, multi stability irreversible hysteresis overdamped oscillators	anharmonic oscillators relaxation oscillations predator-prey models van der Pol systems Chaotic systems	systems of non-linear oscillators non-equilibrium thermodynamics protein structure/ function neural networks the cell ecosystems	Nonlinear wave propagation Reaction-diffusion in dissipative systems Turbulent/chaotic flows

adapted from S. Strogatz