Lecture 2: Linear Systems - Part 1
Winter 2016

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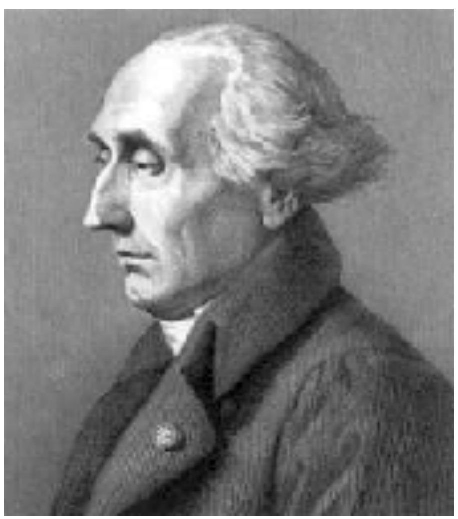
Linear time-invariant systems, Linear ODEs, and the power of decomposability



Joseph-Louis LaGrange
1736 - 1813

Pierre-Simon LaPlace
1749 - 1827

Joseph Fourier
1768 - 1830

So, today we continue to stay with relatively **small-scale linear systems**…

	n = 1	n = 2 or 3	n >> 1	continuum
Linear	exponential growth and decay single step conformational change fluorescence emission pseudo first order kinetics	second order reaction kinetics linear harmonic oscillators simple feedback control sequences of conformational change	electrical circuits molecular dynamics systems of coupled harmonic oscillators equilibrium thermodynamics diffraction, Fourier transforms	Diffusion Wave propagation quantum mechanics viscoelastic systems
Nonlinear	fixed points bifurcations, multi stability irreversible hysteresis overdamped oscillators	anharmonic oscillators relaxation oscillations predator-prey models van der Pol systems Chaotic systems	systems of non-linear oscillators non-equilibrium thermodynamics protein structure/ function neural networks the cell ecosystems	Nonlinear wave propagation Reaction-diffusion in dissipative systems Turbulent/chaotic flows

adapted from S. Strogatz

So, for today, **linear systems analysis**…

The goals will be three-fold:

(1) Understand the origins if simplicity in linear, time-invariant systems (the general regime of most modern engineering)

(2) Understand the principle of decomposability of linear systems…using a simple model of a second order process

(3) Learn a new way to solve differential equations that makes the concepts of linearly and decomposability more intuitive…

Linear Systems Analysis

First, let's consider a change of perspective for today about how we regard basic reactions. We will change from the usual state-centric to a process-centric view:

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A. Folding

$$U \xrightarrow{k} N$$

$$\frac{dU(t)}{dt} = -k\,U(t)$$

$$U(t) = U_0\, e^{-kt}$$

$$\Rightarrow \quad U(t) \rightarrow \boxed{h(t)} \rightarrow N(t)$$

- $U(t)$: the input fn. $t=0$: A_0
- $h(t)$: the system fn
  - in this case, a Poisson, or single exp process.
- $N(t)$: the output response.

Linear Systems Analysis

First, let's consider a change of perspective for today about how we regard basic reactions. We will change from the usual <u>state</u>-centric to a <u>process</u>-centric view:

B. Conf Δ

$$h\nu \rightsquigarrow R \xrightarrow{\alpha} M \quad \Rightarrow \quad \delta(t) \rightarrow \boxed{h(t)} \rightarrow m(t)$$

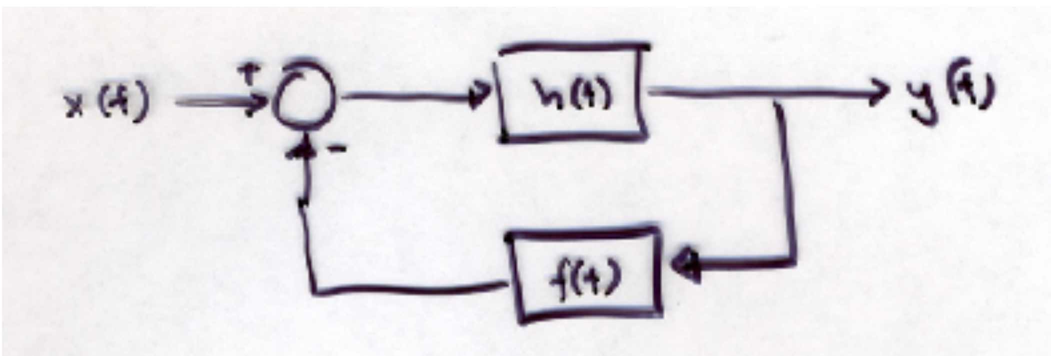
$t=0,\ R_0$

C. serial conf Δ

$$h\nu \rightsquigarrow R \rightarrow I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow M \quad \Rightarrow \quad \delta(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow \dots$$

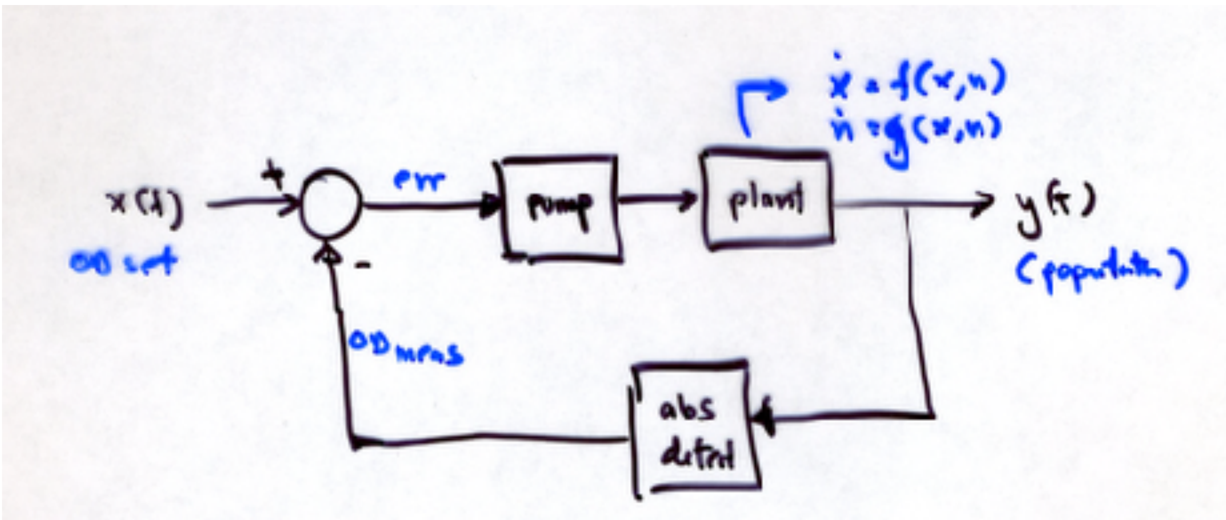
more generally, could be quite complex circuits ....

First, let's consider a change of perspective for today about how we regard basic reactions. We will change from the usual <u>state</u>-centric to a <u>process</u>-centric view:



…this is a basic **feedback control circuit**; a function of the output is fed back to the input to clamp output levels.

First, let's consider a change of perspective for today about how we regard basic reactions. We will change from the usual state-centric to a process-centric view:

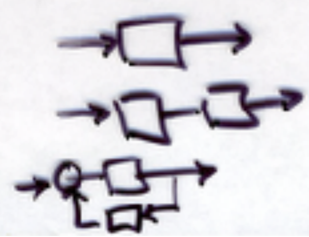


…for example, a system for constant-density growth of micro-organisms (a so-called "**turbidostat**")

Linear Systems Analysis

How can we understand/predict the behavior of these systems? Well, if the systems are linear and time invariant, there is a powerful theory

So...
1. definitions
2. properties of linearity and time invariance
3. convolution theorem
4. The Laplace transform method [for simplicity and intuition]
5. Three examples :



first order

second order

simple feedback system.

…so the **theory of LTI systems**.

Part 1: Linear Time-Invariant Systems

A theme will be to understand the "simplicity" of linear, time-invariant systems. What does "linearity" and "time-invariance" mean exactly?

What does "**linearity**" and "time-invariance" mean exactly?

Linearity implies a principle called superposition: If $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the response to $x_2(t)$, then:

① $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$ [additivity]

② $a \cdot x_1(t) \longrightarrow a \cdot y_1(t)$ [scaling or homogeneity]

What does "**<u>linearity</u>**" and "time-invariance" mean exactly?

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So... if input is

$$x(t) = \sum_k a_k x_k(t) = a_1 x_1(t) + a_2 x_2(t) + \cdots$$

output will be:

$$y(t) = \sum_k a_k y_k(t) = a_1 y_1(t) + a_2 y_2(t) + \cdots$$

This is superposition ... the output is a weighted sum of responses to independent inputs.

What does “linearity” and “**time-invariance**” mean exactly?

Time invariance is just that .... the system behaves the same over time. So...

(1) the response to an input is the same regardless of time when the input comes or...

(2) If $x(t) \rightarrow y(t)$ then..

$$x(t-\tau) \rightarrow y(t-\tau)$$

What does "linearity" and "**time-invariance**" mean exactly?

Time invariance is just that .... the system behaves the same over time. So...

(1) the response to an input is the same regardless of time when the input comes on...

(2) If $x(t) \to y(t)$ then..

$$x(t-\tau) \to y(t-\tau)$$

These two properties underlie the concept of the convolution integral...

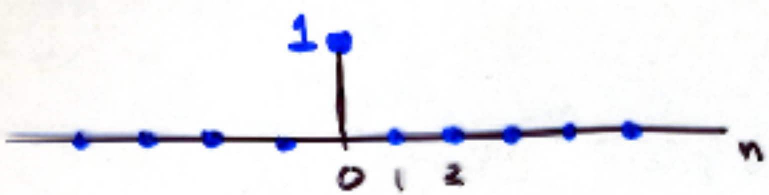
we will take this in steps ...

The Convolution Sum (or Integral, for continuous time systems)

Step 1 : The impulse function . The first concept we need is that of representing any arbitrary input function as a series of scaled , time shifted elementary functions called impulses . . .

In discrete time . . .

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$



$\Rightarrow \delta(n-2)$

The Convolution Sum (or Integral, for continuous time systems)

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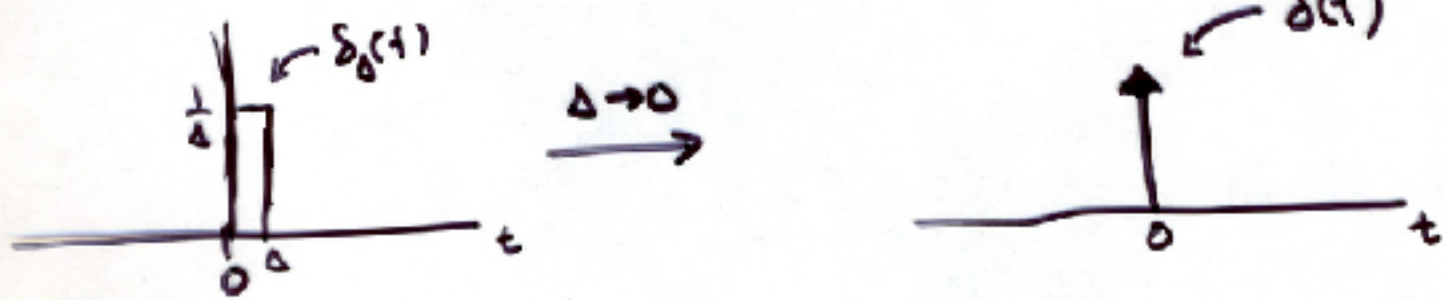
In continuous time ...



$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

Step 2 An input function ...

Again to start, in discrete time ...

$$x(n) = \ldots + x(-3)\,\delta(n+3) + x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + \ldots$$

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$=$

$x(-3)\,\delta(n+3)$

$+$

$x(-2)\,\delta(n+2)$

$+$

$x(-1)\,\delta(n+1)$

$+$

$x(0)\,\delta(0)$

$+$

$\vdots$

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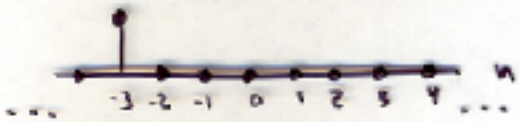
=



$x(-3)\,\delta(n+3)$

+

$x(-2)\,\delta(n+2)$

+

$x(-1)\,\delta(n+1)$

+

$x(0)\,\delta(0)$

+

⋮

or ...

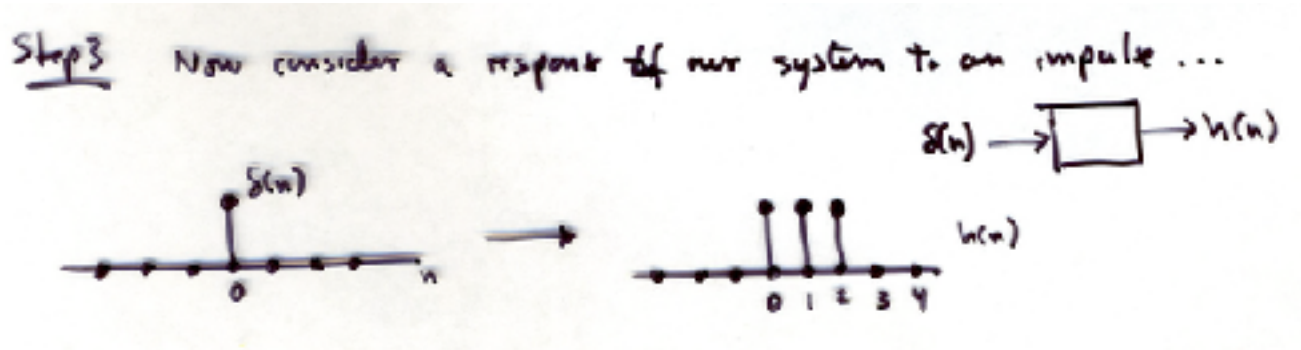
$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

The Convolution Sum (or Integral, for continuous time systems)

<u>Step 3</u> Now consider a response of our system to an impulse ...

This defines the so-called "impulse response" of our system….the output due to an impulse stimulus.

The Convolution Sum (or Integral, for continuous time systems)

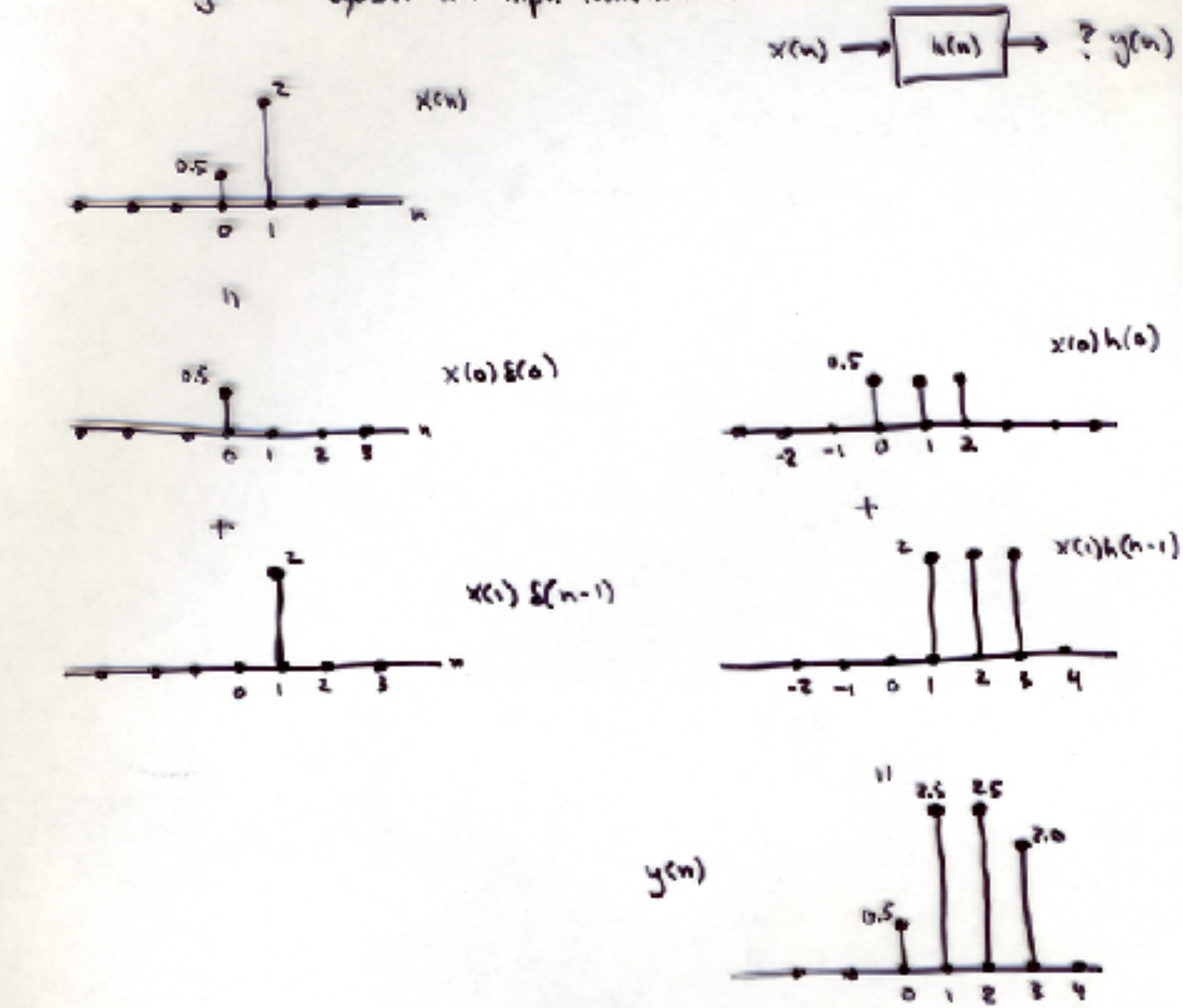


This defines the so-called “impulse response” of our system….the output due to an impulse stimulus.

What then will be there system response to some arbitrary function?

The Convolution Sum (or Integral, for continuous time systems)

Step 4 Now we give our system an input function ... 6



so...

$$y(n) = x(0)h(n) + x(1)\,h(n-1)$$

The Convolution Sum (or Integral, for continuous time systems)

so...

$$y(n) = x(0)h(0) + x(1)h(n-1)$$

Here, we used both the principle of linearity and time invariance !

→ the scaled output to each input impulse was summed

→ the response was the same (though scaled) for each time shift.

The Convolution Sum (or Integral, for continuous time systems)

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more generally ...

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

... this is the convolution sum or superposition sum

For LTI systems, this gives the output ($y(n)$) to any arbitrary input ($x(n)$) given knowledge of its impulse response ($h(n)$).

This is a major result of linear systems analysis ... the response of an LTI system is completely defined by its impulse response.

The Convolution Sum (or Integral, for continuous time systems)

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More generally ...

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For LTI systems, this gives the output $(y(n))$ to any arbitrary input $(x(n))$ given knowledge of its impulse response $(h(n))$.

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In continuous time ...

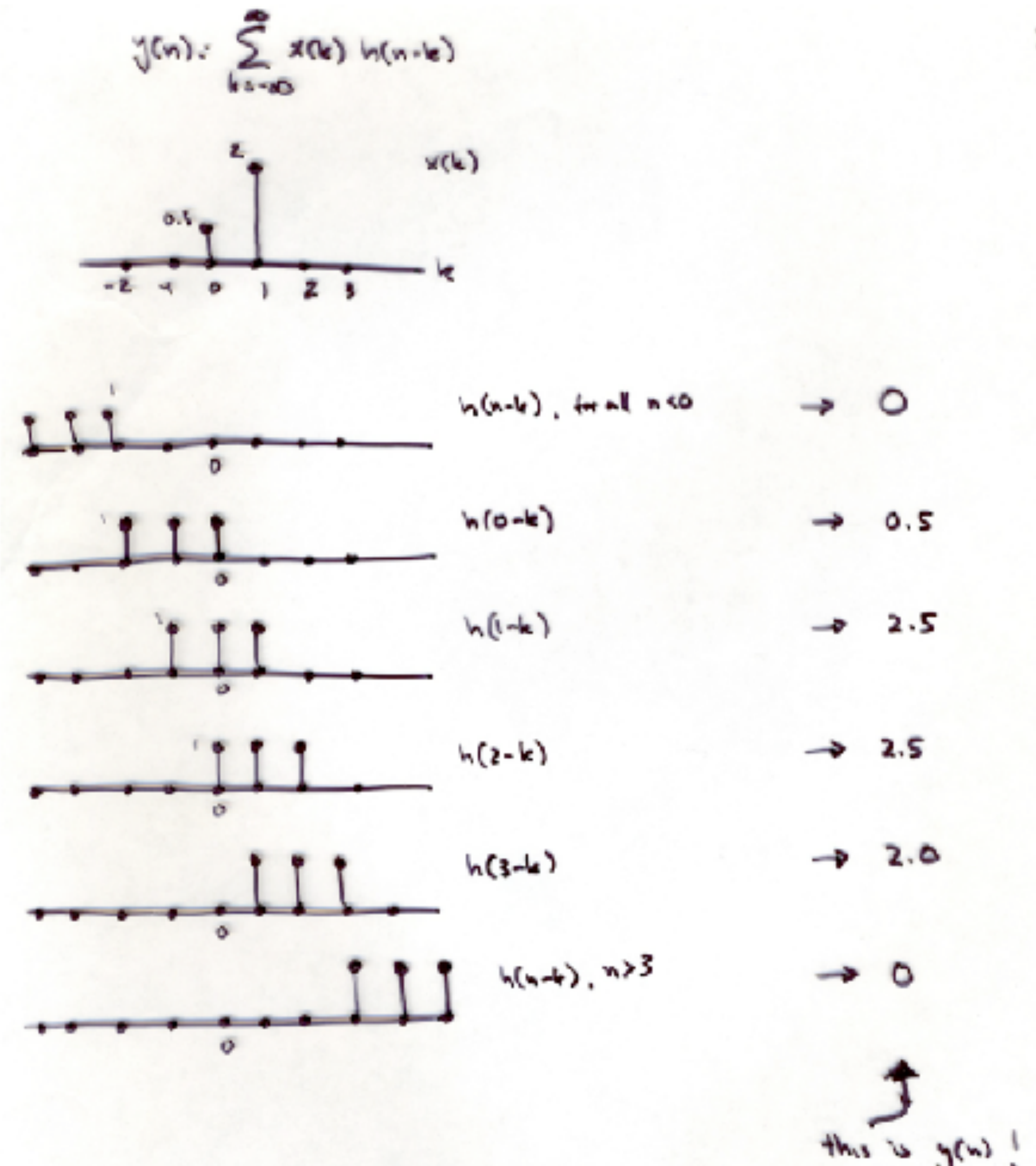
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\boxed{\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \\ y(t) &= x(t) * h(t) \end{aligned}}$$

The convolution integral.

The definition of the convolution operator...and the power of LTI systems....you can completely characterize them by their response to the simplest input...the impulse response!

Now...a simple graphical way of understanding convolution...the sliding of two functions across each other...



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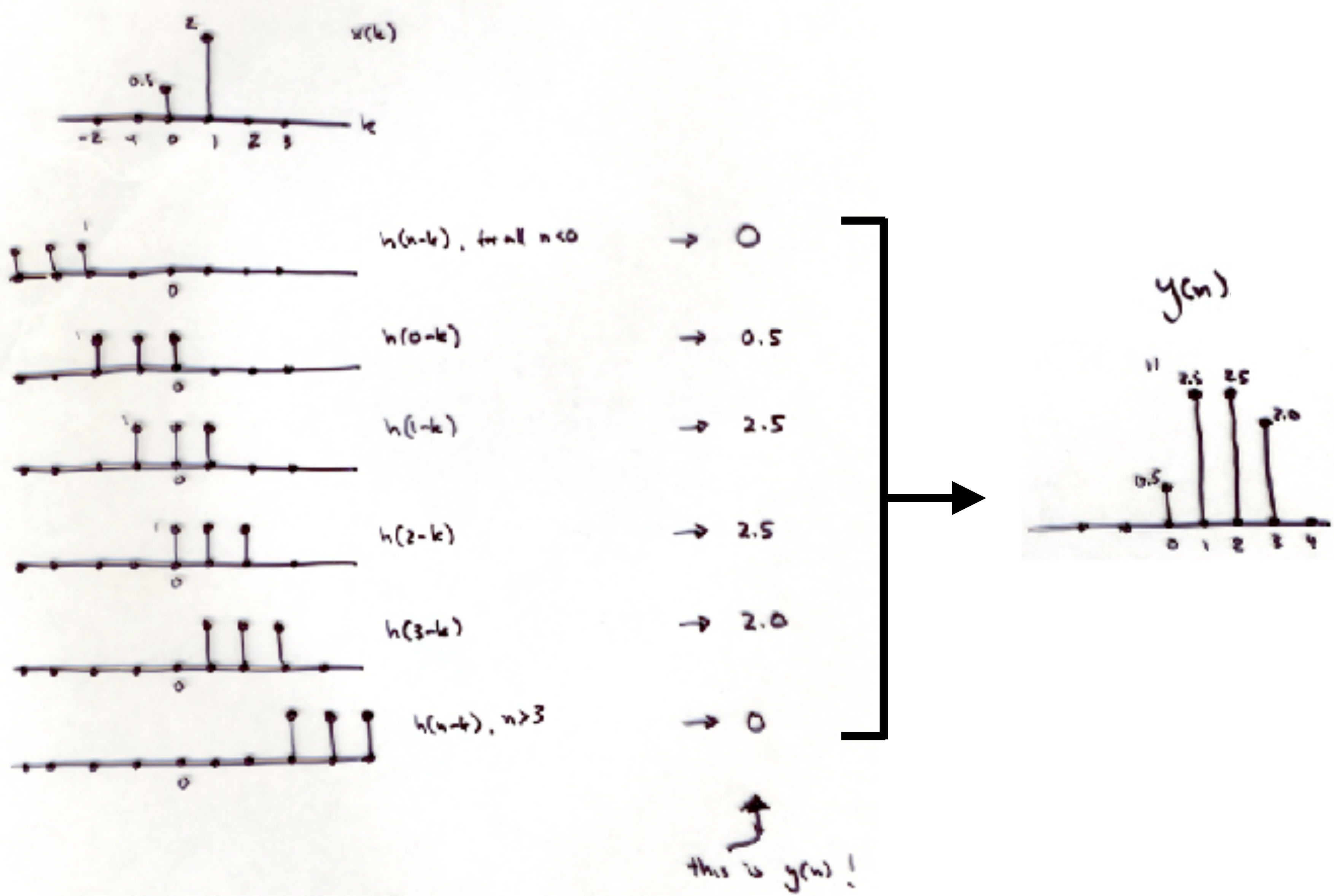


Part 2: reduction of high order linear systems...another powerful property of linearity...

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + \beta x = 0 \qquad \text{[Homogeneous, linear, 2}^{\text{nd}}\text{ order]}$$

Part 2: reduction of high order linear systems...another powerful property of linearity...

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + \beta x = 0 \qquad \text{[Homogeneous, linear, } 2^{nd} \text{ order]}$$

A differential equation is an equation with derivatives in it.

In the general case:

$$g\left[f(t), \frac{df}{dt}, \frac{d^2f}{dt^2}, \ldots, \frac{d^nf}{dt^n}; t\right] = h(t)$$

to refresh…as nicely explained in the mathematics course yesterday

K. A. Reynolds, “Mathematical Foundations”, lecture 1

Part 2: reduction of high order linear systems...another powerful property of linearity...

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + \beta x = 0 \qquad [\text{Homogeneous, linear, } 2^{nd} \text{ order}]$$

But ... the simplest case is a linear, 1st order, homogeneous one:

$$\frac{dy}{dt} = f(t,y)$$

① the equation is of the form:

$$a_0 y + a_1 y' + a_2 y'' + \dots a_n y^n = b$$

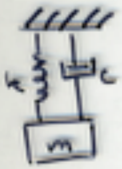
② A first derivative is the highest order derivative in the eqn.

③ All terms contain a y or a derivative of it.

Part 2: reduction of high order linear systems...another powerful property of linearity...

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + \beta x = 0$$ [Homogeneous, linear, 2nd order]

This equation has real physical meaning:



Newtons law says: $\sum F = ma$

$$= m\frac{d^2x}{dt^2}$$

$$\boxed{-kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}}$$

…so this is a second order, linear, homogeneous equation…the equation of motion for **a harmonic oscillator**

Part 2: reduction of high order linear systems...another powerful property of linearity...

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + \beta x = 0 \qquad \text{[Homogeneous, linear, 2}^{\text{nd}}\text{ order]}$$

One way to solve it is to make two new variables:

$$y_1 = x \qquad y_2 = \frac{dx}{dt}$$

Then,

$$\boxed{\frac{dy_1}{dt} = y_2}$$

$$\frac{dy_2}{dt} = -\alpha \frac{dx}{dt} - \beta x$$

$$\boxed{\frac{dy_2}{dt} = -\alpha y_2 - \beta y_1}$$

Note that our second order system just got reduced to **two first order** differential systems!!

The solution to the **linear, first-order, homogeneous** differential equation…

$$\frac{dA}{dt} = -kA$$

$$\frac{1}{A}\,dA = -k\,dt \quad \longrightarrow \text{ separate variables}$$

$$\int_{A_0}^{A} \frac{1}{A}\,dA = \int_0^t -k\,dt \quad \longrightarrow \text{ integrate both sides according to initial conditions and variable range.}$$

$$\ln A \Big|_{A_0}^{A} = -kt \Big|_0^t$$

$$\ln A - \ln A_0 = -kt \quad \longrightarrow \text{ basic algebra}$$

$$\ln\left[\frac{A}{A_0}\right] = -kt$$

$$\boxed{A = A_0 e^{-kt}}$$

This is the analytic solution.

the initial conditions….
$A(0) = A_0$

again…as explained in the mathematics course yesterday

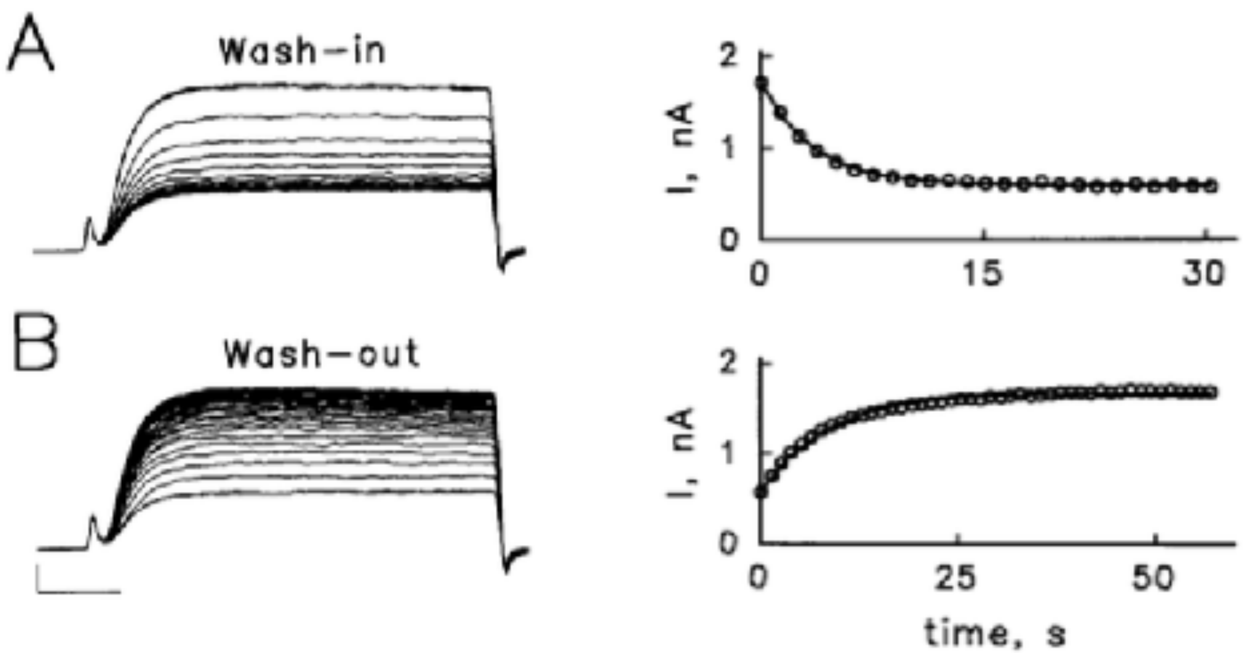
K. A. Reynolds, “Mathematical Foundations”, lecture 1

For example, both binding and dissociation reactions for bimolecular interaction at the pseudo-first order limit is well described by a first-order process…

$$R \underset{k_{-1}}{\overset{k_1 L}{\rightleftharpoons}} LR$$

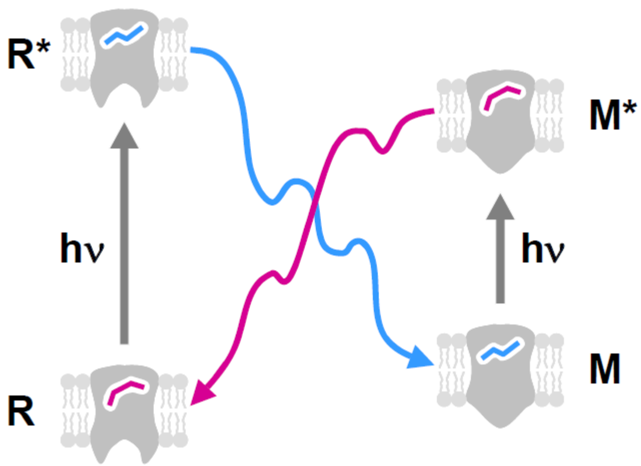
$$R(t) = R_0 e^{-k_{app} t} \quad \text{where} \quad k_{app} = k_1 L_0$$

$$LR(t) = LR_0 e^{-k_{-1} t}$$



Goldstein and Miller, Biophys. J 65, p.1613

What about a **second order system**…the case of multistep conformational change…



Rhodopsin

$$R \xrightarrow{\delta(t)} A \xrightarrow{k_1} B \xrightarrow{k_2} M$$

i) $\frac{dA}{dt} = -k_1 A$

ii) $\frac{dB}{dt} = k_1 A - k_2 B$

$A(0) = A_0$

$B(0) = 0$

…a **system** of differential equations. What is the solution for **A(t)** and **B(t)**?

What about a **second order system**…the case of multistep conformational change…

$$\text{i)} \quad \frac{dA}{dt} = -k_1 A \qquad\qquad A(0) = A_0$$

$$\text{ii)} \quad \frac{dB}{dt} = k_1 A - k_2 B \qquad\qquad B(0) = 0$$

…the solutions:

$$A(t) = A_0 e^{-kt}$$

$$B(t) = \frac{A_0 k_1}{k_2 - k_1}\left[e^{-k_1 t} - e^{-k_2 t} \right]$$

…as demonstrated in the mathematics course yesterday

The solution to the **linear, first-order, inhomogeneous** differential equation…

$$\frac{dB}{dt} = -k_2 B + k_1 A \quad , \text{ where } \quad A = A_0 e^{-k_1 t}$$

$$B(0) = 0$$

The solution is going to be a sum of the homogeneous solution and the particular solution to the specific input.

$$B(t) = \underbrace{B_p(t)}_{\text{the particular solution.}} + \underbrace{B_h(t)}_{\text{the homogeneous solution } \frac{dB}{dt} = -k_2 B}$$

Now, $B_p(t)$ is going to look like the input. So…

$B_p(t) = C A_0 e^{-k_1 t}$. All we need to figure out is what is C.

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The solution to the **linear, first-order, inhomogeneous** differential equation…

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bunch of algebra, using initial conditions…

$$B(t) = \frac{A_0 k_1}{k_2 - k_1}\left[e^{-k_1 t} - e^{-k_2 t} \right]$$

Laplace transforms…

An approach to solving such equations that leads to **some important intuition** about linear systems…

Laplace transforms…

Given any continuously differentiable function $f(t)$, we define the Laplace transform of $f(t)$:

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$

So… this is a transformation of $f(t)$ so that:

$$f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)$$

Why should we do this transformation?

Laplace transforms…

Good reasons for the $\mathcal{L}$:

① Solving differential equations can be much easier, and often with just lookup tables.

② Initial conditions are carried in the process of the solution.

③ With more study, you will learn that they lead to very intuitive forms of differential equations.

The Laplace Transform Method...

Some useful Laplace transforms:

① $f(t) = e^{-kt}$; Then...

$$\mathcal{L}\{f(t)\} =$$

The Laplace Transform Method...

Some useful Laplace transforms:

① $f(t) = e^{-kt}$; Then...

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-kt} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(k+s)t}\, dt$$

$$= -\frac{1}{k+s} e^{-(k+s)t} \Big|_0^\infty$$

$$= \frac{1}{s+k}$$

The Laplace Transform Method...

Some useful Laplace transforms:

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$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-kt} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(k+s)t}\, dt$$

$$= -\frac{1}{k+s} e^{-(k+s)t} \Big|_0^\infty$$

$$= \frac{1}{s+k}$$

So...

$f(t)$	$F(s)$
e^{-kt}	$\frac{1}{s+k}$

The Laplace Transform Method...

(2) $f(t) = t$; Then...

$$F(s) = \int_0^{\infty} t\, e^{-st}\, dt$$

The Laplace Transform Method...

(2) $f(t) = t$; Then...

$$F(s) = \int_0^\infty t e^{-st}\,dt$$

To solve this, we use the old integrate by parts rule:

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

In our case, set:

$$u = t$$
$$dv = e^{-st}dt$$

then:

$$\int_0^\infty t e^{-st}\,dt = -\frac{t}{s}e^{-st}\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\,dt$$

$$= 0 - \left(\frac{1}{s^2}e^{-st}\right)\Big|_0^\infty$$

$$= \frac{1}{s^2}$$

The Laplace Transform Method...

2 $f(t) = t$; Then...

$$F(s) = \int_0^{\infty} t e^{-st} \, dt$$

So ...

$f(t)$	$F(s)$
t	$\frac{1}{s^2}$

The Laplace Transform Method...

(3) Now .. $f(t) = \frac{df}{dt}$

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_0^\infty \frac{df}{dt} e^{-st} dt$$

Again, integrate by parts : $u = e^{-st}$

$dv = \frac{df}{dt} dt$

$$\int_0^\infty \frac{df}{dt} e^{-st} dt = f(t) e^{-st} \Big|_0^\infty - \int_0^\infty f(t)(-s e^{-st})\, dt$$

$$= -f(0) + s \int_0^\infty f(t) e^{-st} dt$$

$$= -f(0) + s F(s)$$

So ...

$f(t)$	$F(s)$
$\frac{df}{dt}$	$sF(s) - f(0)$

The Laplace Transform Method...

and . . . w/o proof

4

$f(t)$	$F(s)$
$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$

The Laplace Transform Method...

ok... using all this, we go back to our differential equation:

$$\frac{dA}{dt} = -kA \qquad , \quad A(0) = A_0$$

$$\mathcal{L}\left\{\frac{dA}{dt}\right\} = \mathcal{L}\left\{-kA\right\}$$

$$sA(s) - A(0) = -k\,A(s)$$

$$sA(s) + kA(s) = A_0$$

$$\boxed{A(s) = \frac{A_0}{s+k}}$$

The transform solution.

The Laplace Transform Method...

To find our solution, we just take the inverse Laplace transform:

$$A(s) = \frac{A_0}{s+k}$$

$$\mathcal{L}^{-1}(A(s)) = \mathcal{L}^{-1}\left[\frac{A_0}{s+k}\right]$$

The Laplace Transform Method...

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$$\mathcal{L}^{-1}(A(s)) = \mathcal{L}^{-1}\left[\frac{A_0}{s+k}\right]$$

remember…

So...

$f(t)$	$F(s)$
e^{-kt}	$\frac{1}{s+k}$

The Laplace Transform Method...

To find our solution, we just take the inverse Laplace transform:

$$A(s) = \frac{A_0}{s+k}$$

$$\mathcal{L}^{-1}(A(s)) = \mathcal{L}^{-1}\left[\frac{A_0}{s+k}\right]$$

$$\boxed{A(t) = A_0 e^{-kt}}$$

Note that we didn't integrate anything! We just did algebra.

In Laplace transform space (F(s)), differentiation and integration become just a matter of doing algebra and finding the inverse transform...

The Laplace Transform Method...

What about the more complicated inhomogeneous first order equation?

$$\frac{dB}{dt} = -k_2 B + k_1 A \quad , \quad \text{where} \quad A = A_0 e^{-k_1 t}$$

$$B(0) = 0$$

The solution to the **linear, first-order, inhomogeneous** differential equation…
the old way…

$$\frac{dB}{dt} = -k_2 B + k_1 A \quad , \text{ where } \quad A = A_0 e^{-k_1 t}$$

$$B(0) = 0$$

The solution is going to be a sum of the homogeneous solution and the particular solution to the specific input.

$$B(t) = \underbrace{B_p(t)}_{\text{the particular solution.}} + \underbrace{B_h(t)}_{\text{the homogeneous solution } \frac{dB}{dt} = -k_2 B}$$

bunch of algebra, using initial conditions…

$$B(t) = \frac{A_0 k_1}{k_2 - k_1}\left[e^{-k_1 t} - e^{-k_2 t}\right]$$

K. A. Reynolds, "Mathematical Foundations", lecture 1

By the Laplace transform approach...

$$\frac{dB}{dt} = -k_2 B + k_1 A$$

$$\frac{dB}{dt} + k_2 B = k_1 A_0 e^{-k_1 t}$$

$$\mathcal{L}\left\{\frac{dB}{dt}\right\} + k_2 \mathcal{L}\{B\} = k_1 A_0 \mathcal{L}\left\{e^{-k_1 t}\right\}$$

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$$\mathcal{L}\left\{\frac{dB}{dt}\right\} + k_2 \mathcal{L}\{B\} = k_1 A_0 \mathcal{L}\{e^{-k_1 t}\}$$

$$s B(s) - B(0) + k_2 B(s) = k_1 A_0 \left[\frac{1}{s+k_1}\right]$$

$B(0)=0$, so ...

$$B(s)[s+k_2] = k_1 A_0 \left[\frac{1}{s+k_1}\right]$$

$$\boxed{B(s) = \frac{k_1 A_0}{(s+k_1)(s+k_2)}}$$

The Transform solution

By the Laplace transform approach...

$$\frac{dB}{dt} = -k_2 B + k_1 A$$

$$\frac{dB}{dt} + k_2 B = k_1 A_o e^{-k_1 t}$$

$$\mathcal{L}\left\{\frac{dB}{dt}\right\} + k_2 \mathcal{L}\{B\} = k_1 A_o \mathcal{L}\left\{e^{-k_1 t}\right\}$$

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$$\boxed{B(s) = \frac{k_1 A_o}{(s+k_1)(s+k_2)}}$$ The Transform solution

$$\mathcal{L}^{-1}\left\{B(s)\right\} = k_1 A_o \mathcal{L}^{-1}\left\{\frac{1}{(s+k_1)(s+k_2)}\right\}$$ * see table

By the Laplace transform approach...

636 INTEGRAL TRANSFORMS

A Short Table of Laplace Transforms

	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\,dt$	
L1	1	$\dfrac{1}{p}$	$\operatorname{Re} p > 0$
L2	e^{-at}	$\dfrac{1}{p+a}$	$\operatorname{Re}(p+a) > 0$
L3	$\sin at$	$\dfrac{a}{p^2+a^2}$	$\operatorname{Re} p > \lvert\operatorname{Im} a\rvert$
L4	$\cos at$	$\dfrac{p}{p^2+a^2}$	$\operatorname{Re} p > \lvert\operatorname{Im} a\rvert$
L5	$t^k,\ k > -1$	$\dfrac{k!}{p^{k+1}}$ or $\dfrac{\Gamma(k+1)}{p^{k+1}}$	$\operatorname{Re} p > 0$
L6	$t^k e^{-at},\ k > -1$	$\dfrac{k!}{(p+a)^{k+1}}$ or $\dfrac{\Gamma(k+1)}{(p+a)^{k+1}}$	$\operatorname{Re}(p+a) > 0$
L7	$\dfrac{e^{-at} - e^{-bt}}{b-a}$	$\dfrac{1}{(p+a)(p+b)}$	$\operatorname{Re}(p+a) > 0$ and
L8	$\dfrac{ae^{-at} - be^{-bt}}{a-b}$	$\dfrac{p}{(p+a)(p+b)}$	$\operatorname{Re}(p+b) > 0$

Lookup inverse transforms...

By the Laplace transform approach...

$$\frac{dB}{dt} = -k_2 B + k_1 A$$

$$\frac{dB}{dt} + k_2 B = k_1 A_0 e^{-k_1 t}$$

$$\mathcal{L}\left\{\frac{dB}{dt}\right\} + k_2 \mathcal{L}\{B\} = k_1 A_0 \mathcal{L}\{e^{-k_1 t}\}$$

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$$\boxed{B(s) = \frac{k_1 A_0}{(s+k_1)(s+k_2)}}$$

The Transform solution

$$\mathcal{L}^{-1}\{B(s)\} = k_1 A_0 \mathcal{L}^{-1}\left\{\frac{1}{(s+k_1)(s+k_2)}\right\}$$

* see table

$$B(t) = k_1 A_0 \left[\frac{e^{-k_1 t} - e^{-k_2 t}}{k_2 - k_1}\right]$$

$$= \frac{A_0 k_1}{k_2 - k_1}\left[e^{-k_1 t} - e^{-k_2 t}\right]$$

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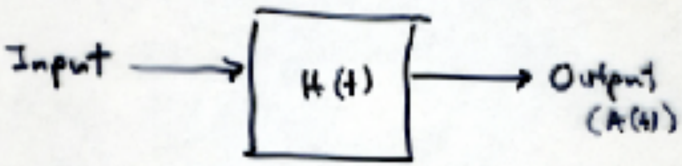
$$= \frac{A_0 k_1}{k_2 - k_1}\left[e^{-k_1 t} - e^{-k_2 t}\right]$$

Much simpler approach...

Part 3: Back to **decomposability of linear systems**…

Another way of thinking about it is that the first order equation is the response of a first order system to an "impulse" stimulus:

Case I: $A = A_0 e^{-kt}$



Input $= \delta(t) A_0$

$H(t) = e^{-kt}$

Output $= A(t)$

But…also an approach to "see" the reduction of a high-order system to a combination of first-order systems...

Another way of thinking about A is that the first order equation is the response of a first order system to an "impulse" stimulus:

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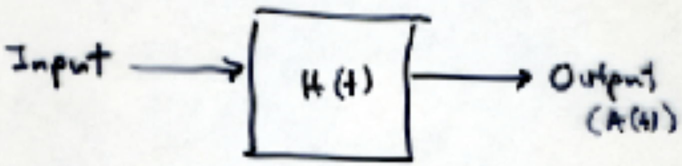


$$\text{Input} = \delta(t)\, A_0$$
$$H(t) = e^{-kt}$$
$$\text{Output} = A(t)$$

In keeping with our process-centric view, this of a first order system as a "process" that converts an input into an output.

The process has a characteristic function…the so-called "transfer function". It is fundamentally defined by the **response to an impulse input**.

Another way of thinking about it is that the first order equation is the response of a first order system to an "impulse" stimulus:

Case I: $A = A_0 e^{-kt}$

$$\text{Input} \longrightarrow \boxed{H(t)} \longrightarrow \text{Output } (A(t))$$

$\text{Input} = \delta(t)\, A_0$

$H(t) = e^{-kt}$

$\text{Output} = A(t)$

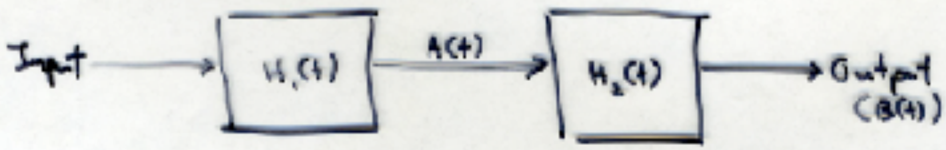
H(t) is sometimes called "a transfer function". It relates the output to the input by this equation:

$$\boxed{\begin{aligned} \text{Output}(t) &= H(t) * \text{Input}(t) \\ \text{Output}(s) &= H(s) \cdot \text{Input}(s) \end{aligned}}$$

Thus, outputs are <u>predictable</u> from just convolutions of the impulse response with the input function…

Lets check our understanding....

Thus, a second order system is like linking two first order systems together, where the output of the first is the input to the second:



Input $= \delta(t)$

$H_1(t) = e^{-k_1 t}$

$H_2(t) = e^{-k_2 t}$

Output $= B(t)$

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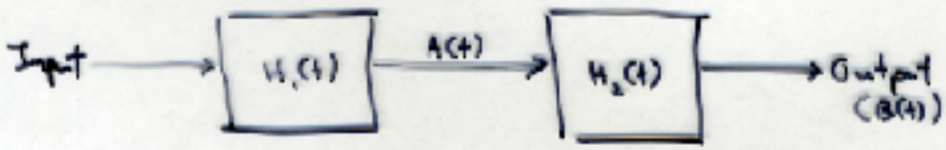

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Output $= B(t)$

So ...

$$A(t) = H_1(t) * \text{Input}(t)$$

$$B(t) = H_2(t) * A(t)$$

$$= H_2(t) * H_1(t) * \text{Input}(t)$$

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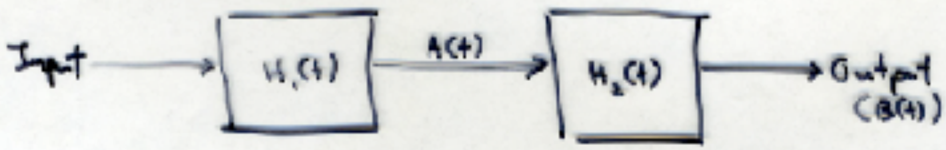

Input $= \delta(t)$

$H_1(t) = e^{-k_1 t}$

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Output $= B(t)$

So ...

$$A(t) = H_1(t) * \text{Input}(t)$$

$$B(t) = H_2(t) * A(t)$$

$$= H_2(t) * H_1(t) * \text{Input}(t)$$

Let's use this to study our two cases...a first order system (single exponential) and our second order system (a double exponential)...

Case I: $A_0\delta(t) \longrightarrow \boxed{H(t)} \longrightarrow A(t)$

$H(t) = e^{-kt}$

$\delta(t)$ = (the impulse function)

We said that:

$$A(t) = H(t) * \delta(t) A_0$$

In Laplace transforms:

$$A(s) = H(s) \cdot \mathcal{L}\{\delta(t)\} A_0 \quad ; \text{ Now } \mathcal{L}\{\delta(t)\} = 1$$

So...

$$A(s) = \frac{A_0}{s+k}$$

And..

$$A(t) = A_0 e^{-kt}$$

So, the first order system can be thought of as a system driven by an impulse stimulus...

Case II:

$$A_0\,\delta(t) \longrightarrow \boxed{H_1(t)} \longrightarrow \boxed{H_2(t)} \longrightarrow B(t)$$

$H_1(t) = e^{-k_1 t}$

$H_2(t) = e^{-k_2 t}$

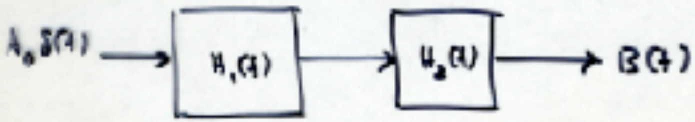
$\delta(t)$ = impulse function

$$B(t) = H_1(t) * H_2(t) * \delta(t)\,A_0$$

$$B(s) = H_1(s) \cdot H_2(s) \cdot A_0$$

So... the second order process of producing $B(t)$ is due to the serial attachment of first order processes.

Case II:



$H_1(t) = e^{-k_1 t}$
$H_2(t) = e^{-k_2 t}$
$\delta(t)$ = impulse function

$$B(t) = H_1(t) * H_2(t) * \delta(t) A_0$$

$$B(s) = H_1(s) \cdot H_2(s) \cdot A_0$$

So... the second order process of producing $B(t)$ is due to the serial attachment of first order processes.

Do you see:

(1) that the first half of the block diagram is just a homogeneous first order differential eqn?

(2) that the second half is just the inhomogeneous first order differential eqn?

$$\boxed{\frac{dA}{dt} = -kA}$$

$$\frac{dB}{dt} = -k_2 B + k_1 A$$

So, a second order system is a series of two first order systems….a basic property of linearity!

This reduction of a high-order system to a combination of first-order systems is a fundamental property of linear systems...

Is this general for any order equation? Yes...

Say we have:

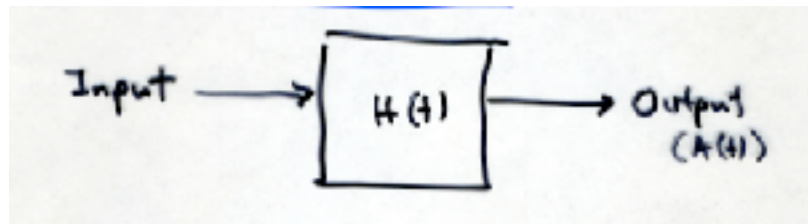
$$\frac{d^n u}{dx^n} = f(t,u) \qquad (n^{th} \text{ order})$$

we can decompose this into a system of n first order eqns:

Definition of y_i	1^{st} order Eqn for y_i
$y_1 = u$	$\frac{dy_1}{dt} = y_2$
$y_2 = \frac{du}{dt}$	$\frac{dy_2}{dt} = y_3$
$y_3 = \frac{d^2u}{dt^2}$	$\frac{dy_3}{dt} = y_4$
$\vdots$	
$y_n = \frac{d^{n-1}u}{dt^{n-1}}$	$\frac{dy_n}{dt} = f(t,u)$

So...linear time-invariant systems are “simple” (not complex) for two reasons:

(1) They have the property that the impulse response fully characterizes their behavior. All responses to more complex inputs are just a convolution of the impulse response (the transfer function) with the input function.



$$\boxed{\begin{aligned} \text{Output}(t) &= H(t) * \text{Input}(t) \\ \text{Output}(s) &= H(s) \cdot \text{Input}(s) \end{aligned}}$$

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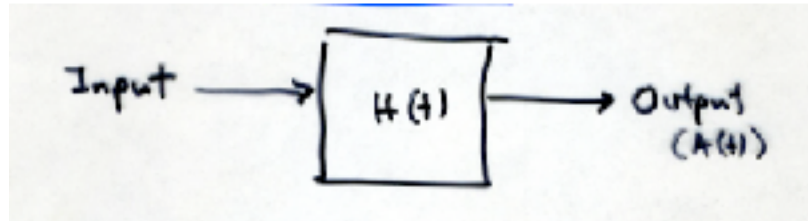
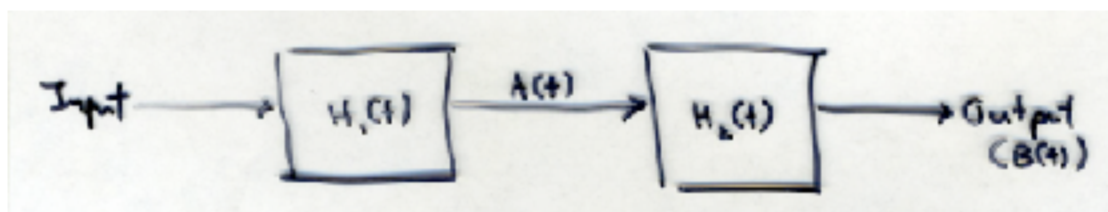


$$\mathrm{Output}(t) = H(t) * \mathrm{Input}(t)$$

$$\mathrm{Output}(s) = H(s) \cdot \mathrm{Input}(s)$$

(2) Higher order systems can always be broken down into a serial process of linked first order systems....



$$A(t) = H_1(t) * \mathrm{Input}(t)$$

$$B(t) = H_2(t) * A(t)$$

$$= H_2(t) * H_1(t) * \mathrm{Input}(t)$$

Next time…a full analysis of **n = 2 linear systems**…and graphical tools

	n = 1	n = 2 or 3	n >> 1	continuum
Linear	exponential growth and decay single step conformational change fluorescence emission pseudo first order kinetics	second order reaction kinetics linear harmonic oscillators simple feedback control sequences of conformational change	electrical circuits molecular dynamics systems of coupled harmonic oscillators equilibrium thermodynamics diffraction, Fourier transforms	Diffusion Wave propagation quantum mechanics viscoelastic systems
Nonlinear	fixed points bifurcations, multi stability irreversible hysteresis overdamped oscillators	anharmonic oscillators relaxation oscillations predator-prey models van der Pol systems Chaotic systems	systems of non-linear oscillators non-equilibrium thermodynamics protein structure/ function neural networks the cell ecosystems	Nonlinear wave propagation Reaction-diffusion in dissipative systems Turbulent/chaotic flows

adapted from S. Strogatz

Next, the obvious **mathematical model**….

$$R \xrightarrow{\delta(t)} A \xrightarrow{k_1} B \xrightarrow{k_2} M$$

$$\frac{dB}{dt} = -k_2 B + k_1 A \quad , \text{ where } \quad A = A_0 e^{-k_1 t}$$

$$B(0) = 0$$

How do we solve this differential equation? Well here is one way….

$$\frac{dB}{dt} = -k_2 B + k_1 A \quad , \text{ where } \quad A = A_0 e^{-k_1 t}$$

$$B(0) = 0$$

We make a proposal….

The solution is going to be a sum of the homogeneous solution and the particular solution to the specific input.

$$B(t) = \underline{B_p(t)} + \underline{B_h(t)}$$

$\rightarrow$ the homogeneous solution $\frac{dB}{dt} = -k_2 B$

$\rightarrow$ the particular solution.

The idea is to think of this system as having two parts to its solution….one that looks like its “natural” response (the <u>homogeneous solution</u>) and one that looks like the input into it (the <u>particular solution</u>). Let’s look at the particular solution first…

$$\frac{dB}{dt} = -k_2 B + k_1 A \quad , \text{ where } \quad A = A_0 e^{-k_1 t}$$

$$B(0) = 0$$

The solution is going to be a sum of the homogeneous solution and the particular solution to the specific input.

$$B(t) = \underline{B_p(t)} + \underline{B_h(t)}$$

$\to$ the homogeneous solution $\frac{dB}{dt} = -k_2 B$

$\to$ the particular solution.

Now, $B_p(t)$ is going to look like the input. So...

10

$B_p(t) = C A_0 e^{-k_1 t}$. All we need to figure out is what is C.

$B(t) = \underline{B_p(t)} + \underline{B_h(t)}$

the homogen solution $\frac{dB}{dt} = -k_2 B$

the particular solution.

Now, $B_p(t)$ is going to look like the input. So...

$B_p(t) = C A_o e^{-k_1 t}$. All we need to figure out is what is C.

$$\frac{dB_p}{dt} + k_2 B_p = k_1 A$$
$$= k_1 A_o e^{-k_1 t}$$

$$-k_1 C A_o e^{-k_1 t} + k_2 C A_o e^{-k_1 t} = k_1 A_o e^{-k_1 t}$$

$$-k_1 C + k_2 C = k_1$$

$$C = \frac{k_1}{k_2 - k_1}$$

So...

$$\boxed{B_p(t) = \frac{k_1 A_o}{k_2 - k_1} e^{-k_1 t}}$$

The particular solution

And now for the “homogeneous solution”...

$$B(t) = \underbrace{B_p(t)}_{\text{the particular solution.}} + \underbrace{B_h(t)}_{\text{the homogeneous solution } \frac{dB}{dt} = -k_2 B}$$

$$B(t) = B_p(t) + B_h(t)$$

$$= \frac{A_0 k_1}{k_2 - k_1} e^{-k_1 t} + D e^{-k_2 t}$$

... Now what is D?

$B(0) = 0$, so ..

$$0 = \frac{A_0 k_1}{k_2 - k_1} + D$$

So, putting it all together...

$$B(t) = B_p(t) + B_h(t)$$

the homogeneous solution $\frac{dB}{dt} = -k_2 B$

the particular solution.

$$B(t) = B_p(t) + B_h(t)$$

$$= \frac{A_0 k_1}{k_2 - k_1} e^{-k_1 t} + D e^{-k_2 t}$$

... Now what is D?

$B(0) = 0$, so ..

$$0 = \frac{A_0 k_1}{k_2 - k_1} + D$$

So, putting it all together...

Thus ...

$$\boxed{B(t) = \frac{A_0 k_1}{k_2 - k_1}\left[e^{-k_1 t} - e^{-k_2 t} \right]}$$

The final solution