Lecture 10: Non-linear Dynamical Systems - Part 2

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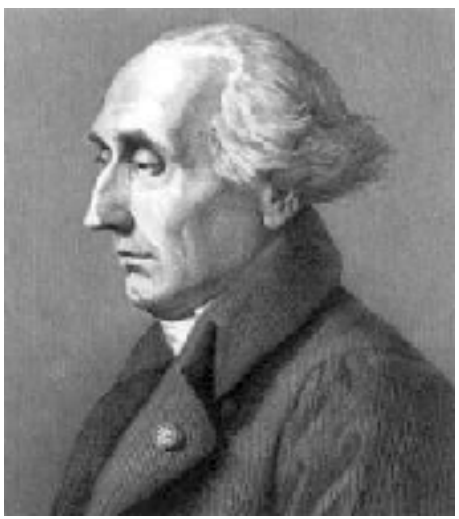
Principles of stability and bifurcation in simple **non-linear** dynamical systems



Joseph-Louis LaGrange
1736 - 1813

Henri Poincare
1854 - 1912

Edward Lorenz
1917-2008

Mitchell Feigenbaum
1944 -

Robert May
1936 -

Albert Libchaber
1934 -

So, today we explore the truly astounding emergent complexity inherent in even simple **non-linear dynamical systems**.

	n = 1	n = 2 or 3	n >> 1	continuum
Linear	exponential growth and decay single step conformational change fluorescence emission pseudo first order kinetics	second order reaction kinetics linear harmonic oscillators simple feedback control sequences of conformational change	electrical circuits molecular dynamics systems of coupled harmonic oscillators equilibrium thermodynamics diffraction, Fourier transforms	Diffusion Wave propagation quantum mechanics viscoelastic systems
Nonlinear	fixed points bifurcations, multi stability irreversible hysteresis overdamped oscillators	anharmonic oscillators relaxation oscillations predator-prey models van der Pol systems Chaotic systems	systems of non-linear oscillators non-equilibrium thermodynamics protein structure/ function neural networks the cell ecosystems	Nonlinear wave propagation Reaction-diffusion in dissipative systems Turbulent/chaotic flows

adapted from S. Strogatz

So, today:

(1) A reminder of the van der Pol oscillator - a small **non-linear system** that illustrates analysis of stability, limit cycle oscillations, and bifurcation.

(2) Concepts of **local linearization** and formal analyses of stability and bifurcation. Examples in the classic van der Pol oscillator, the Fitzhigh-Nagumo model, and perhaps a higher order system.

The **general solution** to second-order linear system…

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

$$\rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\dot{\mathbf{x}} = A\mathbf{x}$$ given $\mathbf{x_0}$ …a vector of initial conditions

$$\downarrow$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x_0}$$

….where **A** is the characteristic matrix. It's properties control all behaviors of the system

Properties of the characteristic matrix…

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{aligned} \tau &= \mathrm{trace}(A) = a + d\,, \\ \Delta &= \det(A) = ad - bc\,. \end{aligned}$$

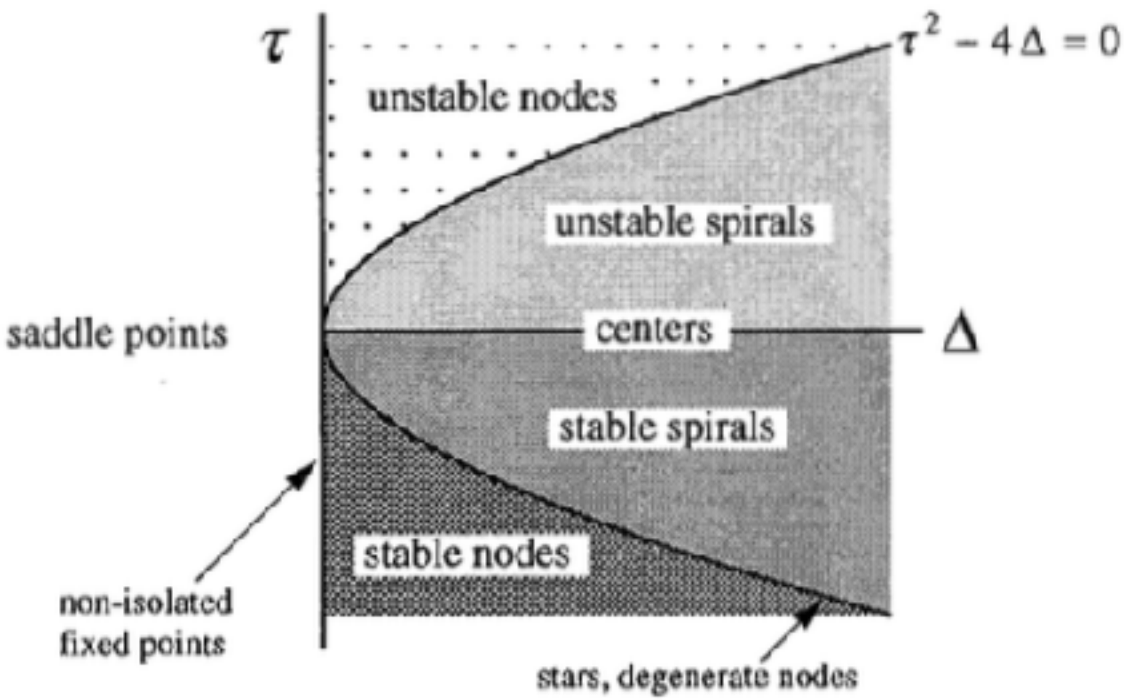
↓

$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

…the **eigenvalues** of **A** are completely determined by the **trace** and **determinant**…

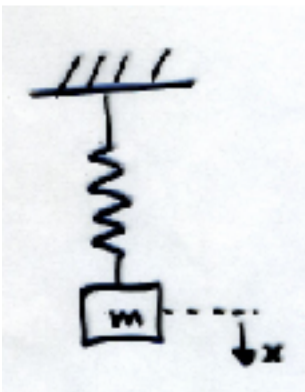
System behaviors: the second order linear case

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$



…stability is determined by the **real part** of the eigenvalue…

Seeing behaviors: the linear harmonic oscillator



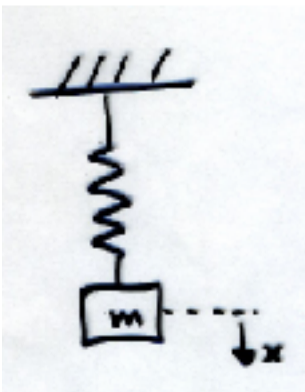
$$m\ddot{x} + kx = 0$$

$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x$$

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

We know how to **analyze the behavior**….right?

Seeing behaviors: the linear harmonic oscillator



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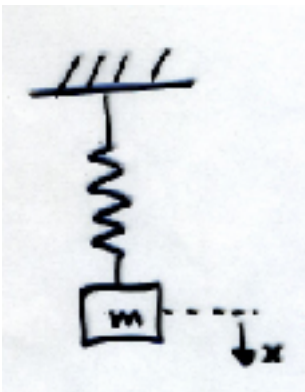
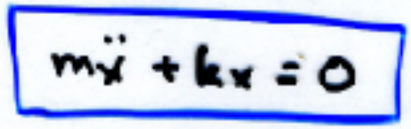
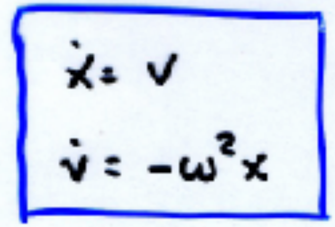
$$\lambda_{1,2} = \frac{1}{2}\left[0 \pm \sqrt{0 - 4\omega^2}\right]$$

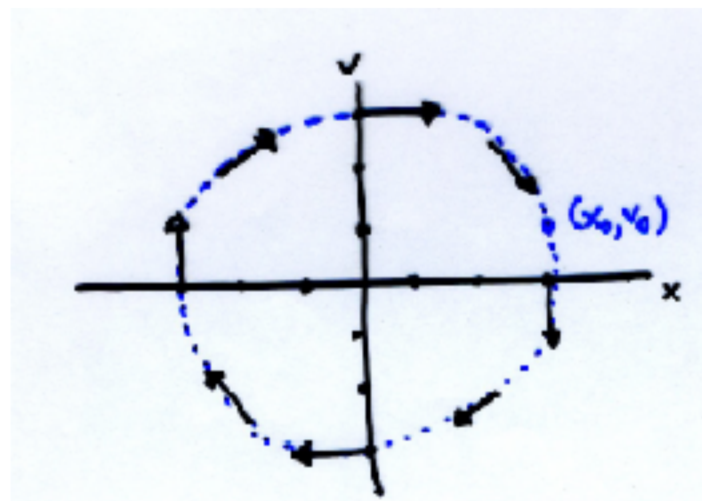
Eigenvalues are pure imaginary….so **centers**!

Seeing behaviors: the linear harmonic oscillator









Eigenvalues are pure imaginary….so **centers**!

And…remember the **system nullclines,** which help us sketch the behavior in the **phase space**…

A **non-linear oscillator**...

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

(the van der Pol oscillator)

Here is the **non-linearity**….with mu controlling the degree of non-linearity.

A **non-linear oscillator...**

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0 \quad (1)$$

$$\ddot{x} + \mu(x^2-1)\dot{x} = \frac{d}{dt}\left(\dot{x} + \mu\left[\frac{1}{3}x^3 - x\right]\right) \quad (2)$$

$$F(x) = \frac{1}{3}x^3 - x \quad (3)$$

$$w = \dot{x} + \mu F(x) \quad (4)$$

So...

$$\dot{w} = \ddot{x} + \mu(x^2-1)\dot{x} = -x \qquad \text{using (4), (2), (1)}$$

$y = \frac{w}{\mu}$. Then...

$$\dot{x} = \mu\left[y - F(x)\right]$$
$$\dot{y} = -\frac{1}{\mu}x$$

Re-writing the equations in a **more intuitive** way....

A **non-linear oscillator**...

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

(the van der Pol oscillator)

$$\dot{x} = \mu\left[y - F(x)\right]$$
$$\dot{y} = -\frac{1}{\mu}x$$

, where

$F(x) = \frac{1}{3}x^3 - x$

$w = \dot{x} + \mu F(x)$

$y = \frac{w}{\mu}$

Now, we compute **fixed points** and **nullclines** and sketch the behavior in the x, y space….

A **non-linear oscillator**...

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$ (the van der Pol oscillator)

↓

$$\dot{x} = \mu\left[y - F(x)\right]$$
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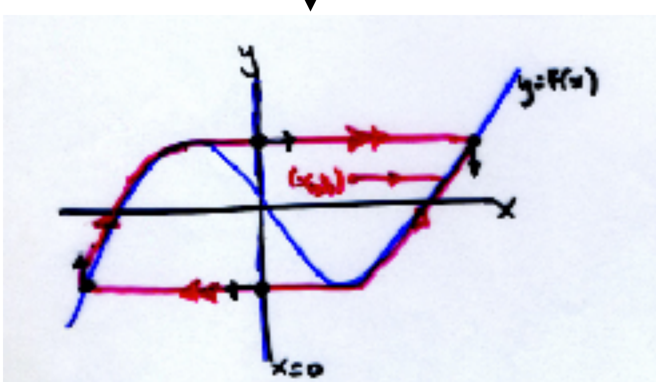

Fixed point at origin. At any non-zero (x, y) the system has a **limit cycle oscillation**…

A **non-linear oscillator...**

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

(the van der Pol oscillator)

$$\dot{x} = \mu[y - F(x)]$$
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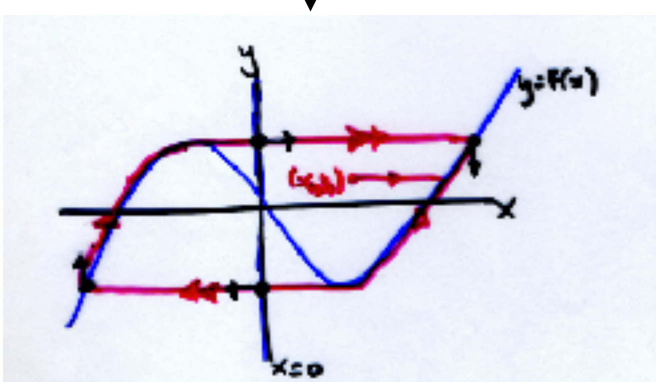

Fixed point at origin. At any non-zero (x, y) the system has a **limit cycle oscillation**…

A closed orbit that another trajectory spirals into as time goes to infinity…

A **non-linear oscillator...**

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

(the van der Pol oscillator)

$$\dot{x} = \mu[y - F(x)]$$
$$\dot{y} = -\frac{1}{\mu}x$$

, where

$$F(x) = \frac{1}{3}x^3 - x$$
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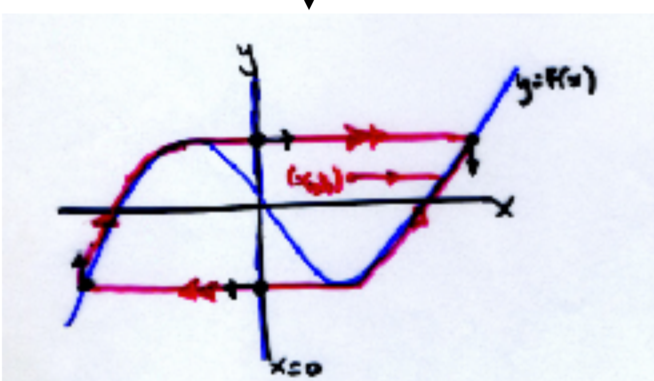

The limit cycle shows the property of a large **divergence of time scales**….

$$y - F(x) \sim O(1) \longrightarrow \begin{array}{l} |\dot{x}| \sim O(\mu) \gg 1 \\ |\dot{y}| \sim O(\mu^{-1}) \ll 1 \end{array}$$

$$y - F(x) \sim O(\mu^{-2}) \longrightarrow \begin{array}{l} |\dot{x}| \sim O(\mu^{-1}) \\ |\dot{y}| \sim O(\mu^{-1}) \end{array}$$

A **non-linear oscillator...**

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$ (the van der Pol oscillator)

$$\dot{x} = \mu[y - F(x)]$$
$$\dot{y} = -\frac{1}{\mu}x$$

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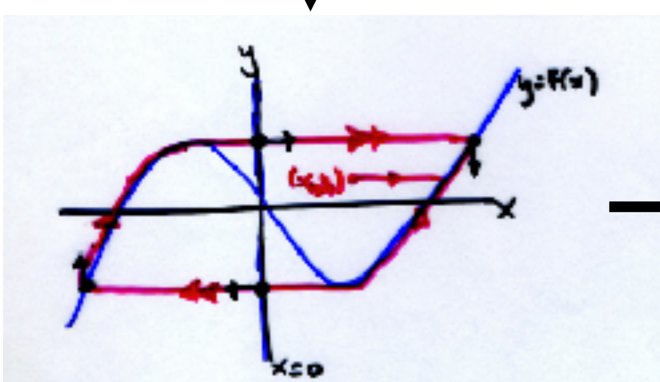
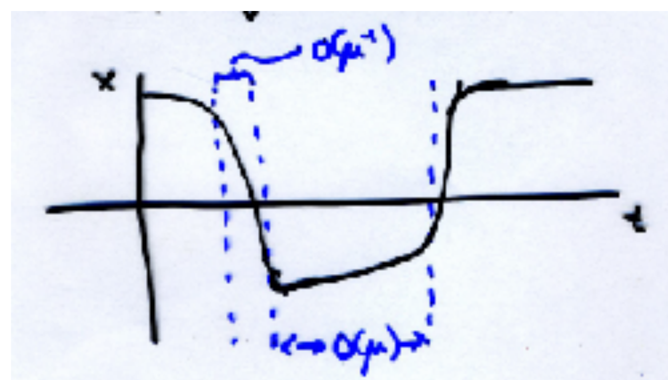
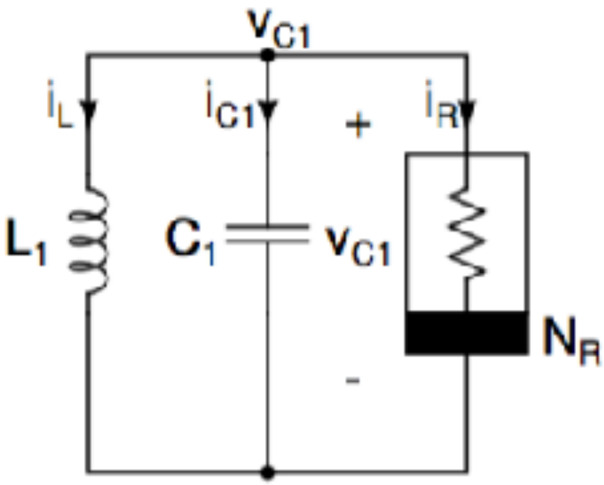


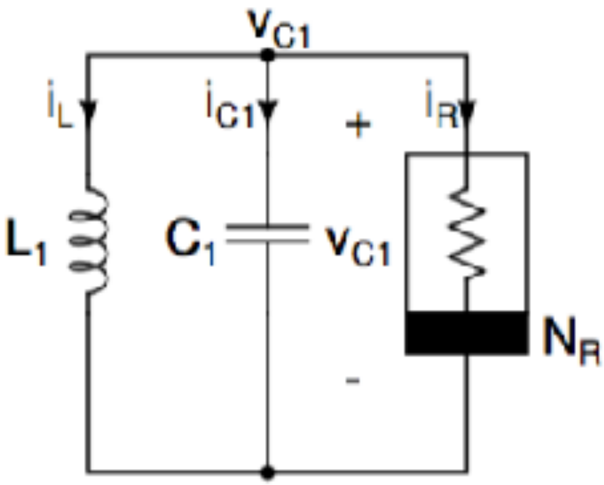


A **non-linear oscillator...**



An electrical **circuit implementation** of the van der Pol oscillator…

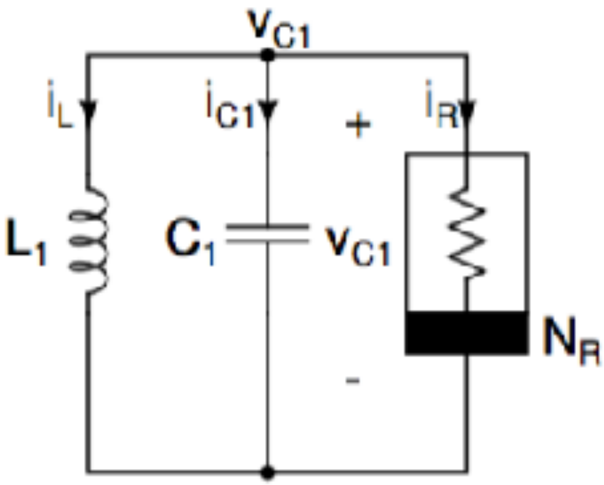
A **non-linear oscillator...**



An electrical **circuit implementation** of the van der Pol oscillator…

a network of elementary **passive** and **active** components

A **non-linear oscillator...**



An electrical **circuit implementation** of the van der Pol oscillator…

a network of elementary **passive** and **active** components

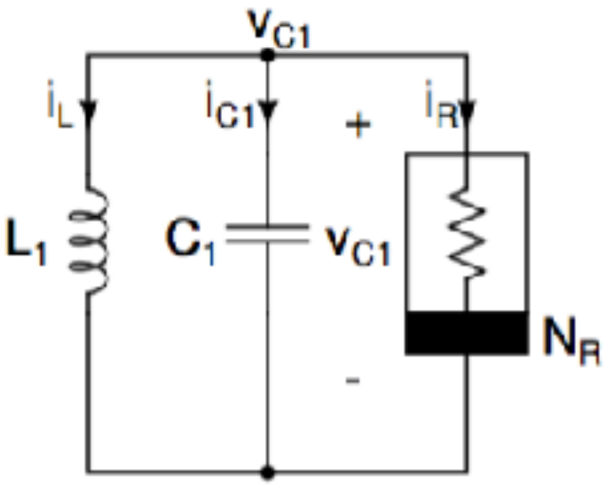
things that use but do not produce energy

$$\Delta V = Ri$$ (resistor)

$$i = C\frac{dV}{dt}$$ (capacitor)

$$\Delta V = L\frac{di}{dt}$$ (inductor)

A **non-linear oscillator...**

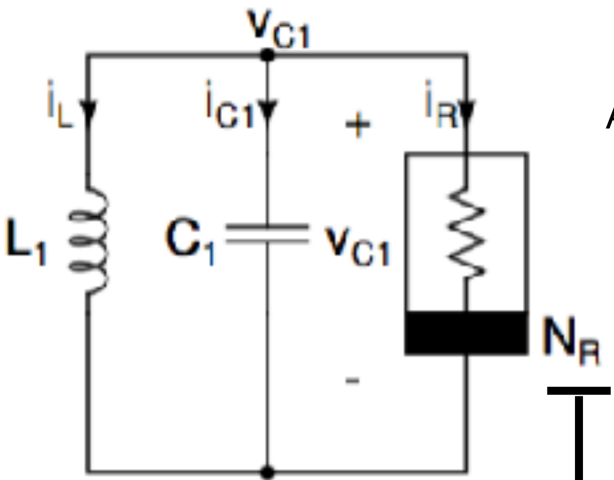


An electrical **circuit implementation** of the van der Pol oscillator…

a network of elementary **passive** and **active** components

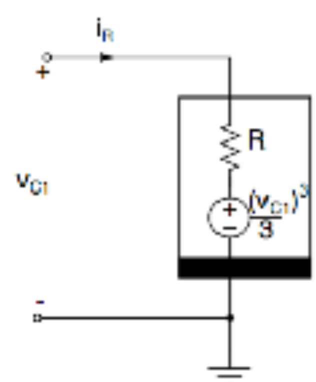
voltage and current sources

A **non-linear oscillator...**



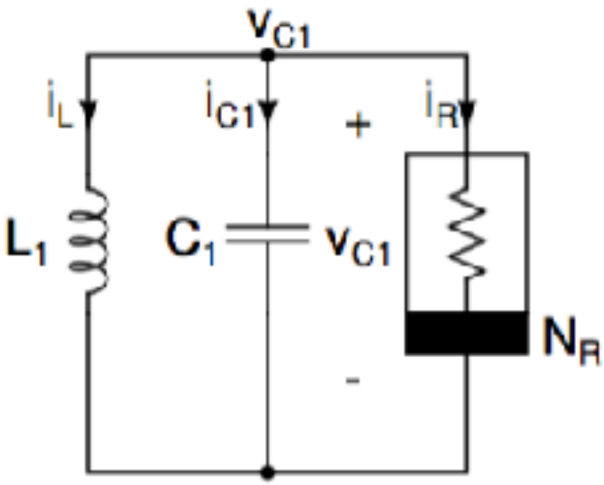
An electrical **circuit implementation** of the van der Pol oscillator…

here, there is also a **non-linear resistor**…with a cubic response function



$$i_R(v_{C1}) = (av_{C1} + bv_{C1}^3)\frac{1}{R}$$

A **non-linear oscillator...**

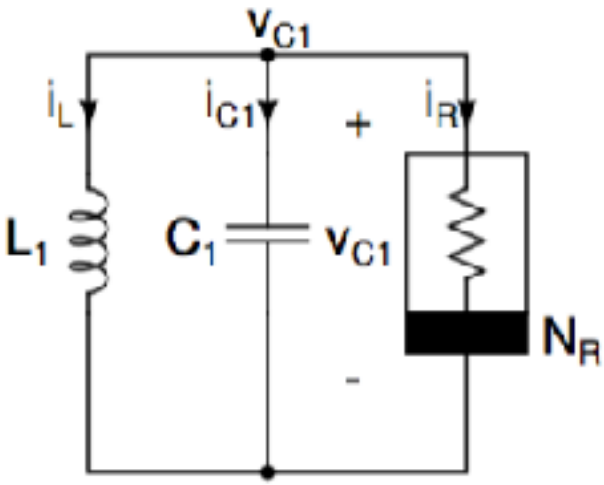


Based on Kirchhoff's voltage law…

$$i_L + i_{C1} + i_R = 0$$, or….

$$\frac{di_L}{dt} + \frac{di_{C1}}{dt} + \frac{di_R}{dt} = 0$$

A **non-linear oscillator...**



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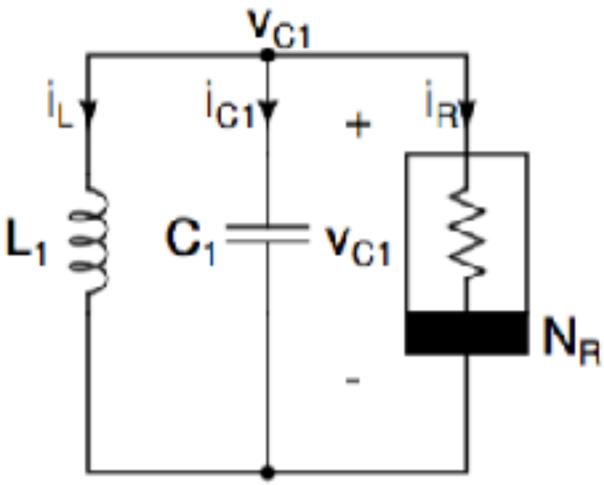
A **non-linear oscillator...**



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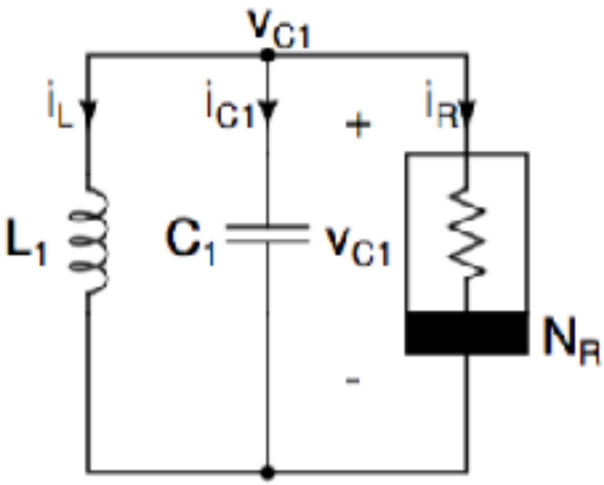
$$\frac{v_{C1}}{L_1} + C_1\frac{d^2v_{C1}}{dt^2} + \frac{di_R}{dv_{C1}}\frac{dv_{C1}}{dt} = 0$$

because of the chain rule…

$$\frac{d(i_R(v_{C1}))}{dt} = \frac{di_R}{dv_{C1}}\frac{dv_{C1}}{dt}$$

A **non-linear oscillator...**



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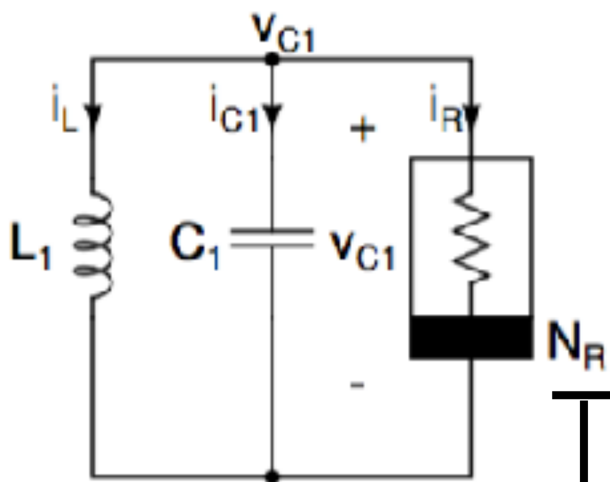
$$\frac{v_{C1}}{L_1} + C_1 \frac{d^2 v_{C1}}{dt^2} + \frac{di_R}{dv_{C1}} \frac{dv_{C1}}{dt} = 0$$

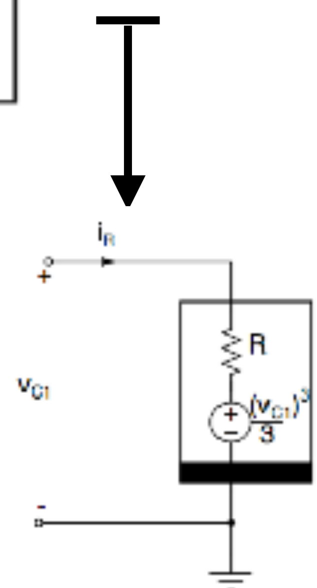
we can simplify this using our basic voltage equation for the non-linear resistor…

$$i_R(v_{C1}) = (av_{C1} + bv_{C1}^3)\frac{1}{R}$$

A **non-linear oscillator...**



An electrical **circuit implementation** of the van der Pol oscillator…

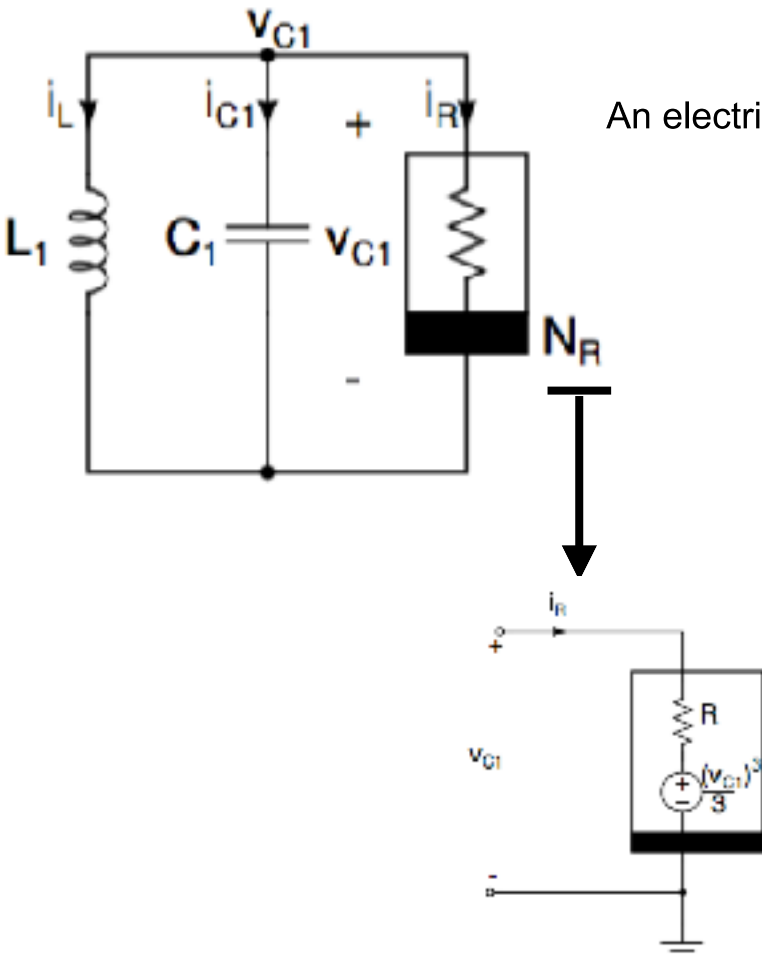


$$i_R(v_{C1}) = (a v_{C1} + b v_{C1}^3)\frac{1}{R}$$

if we choose **a =1** and **b = 1/3**….

A **non-linear oscillator...**

An electrical **circuit implementation** of the van der Pol oscillator…



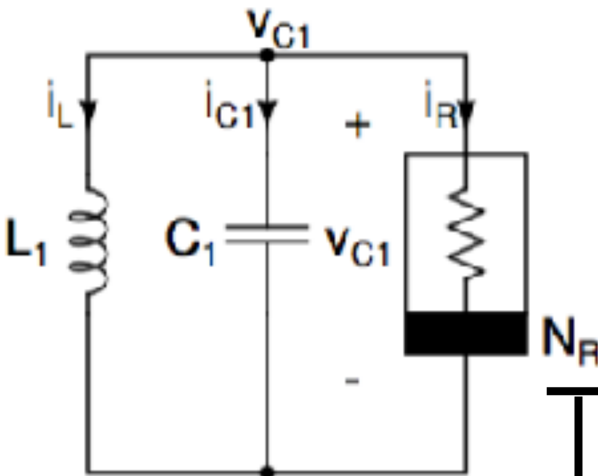
$$i_R(v_{C1}) = (a v_{C1} + b v_{C1}^3)\frac{1}{R}$$

if we choose **a =1** and **b = 1/3**….

$$i_R(v_{C1}) = (-v_{C1} + \frac{1}{3}v_{C1}^3)\frac{1}{R}$$

starting to look like our usual **van der Pol system**…

A **non-linear oscillator...**



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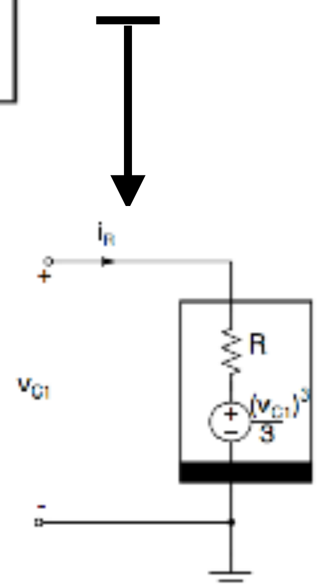

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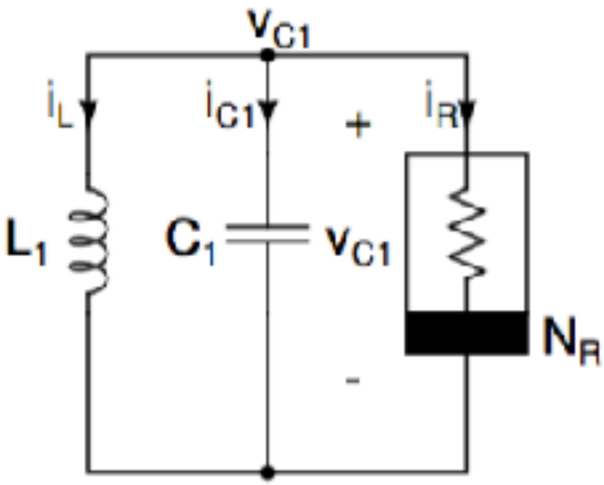
if we choose **a =1** and **b = 1/3**….

$$i_R(v_{C1}) = (-v_{C1} + \frac{1}{3}v_{C1}^3)\frac{1}{R}$$

and, taking the derivative….

$$\frac{di_R(v_{C1})}{dv_{C1}} = (-1 + v_{C1}^2)\frac{1}{R}$$

A **non-linear oscillator...**



Based on **Kirchhoff's voltage law**…

$$i_L + i_{C1} + i_R = 0$$, or….

$$\frac{di_L}{dt} + \frac{di_{C1}}{dt} + \frac{di_R}{dt} = 0$$

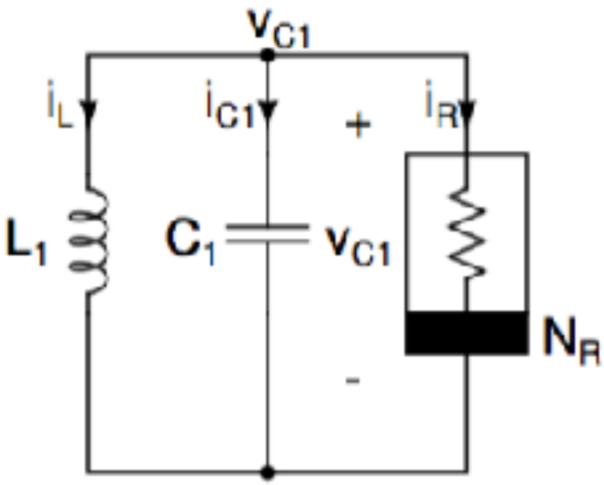
$$\frac{v_{C1}}{L_1} + C_1 \frac{d^2 v_{C1}}{dt^2} + \frac{di_R}{dv_{C1}} \frac{dv_{C1}}{dt} = 0$$

↓

$$C_1 \frac{d^2 v_{C1}}{dt^2} + \left(-1 + v_{C1}^2\right) \frac{1}{R} \frac{dv_{C1}}{dt} + \frac{v_{C1}}{L_1} = 0$$

this is the **differential equation** that controls our system…

A **non-linear oscillator...**



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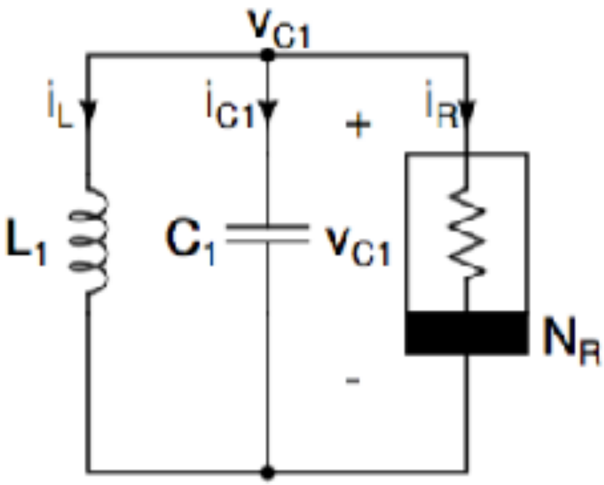
$$\frac{v_{C1}}{L_1} + C_1\frac{d^2v_{C1}}{dt^2} + \frac{di_R}{dv_{C1}}\frac{dv_{C1}}{dt} = 0$$

$$C_1\frac{d^2v_{C1}}{dt^2} + \left(-1 + v_{C1}^2\right)\frac{1}{R}\frac{dv_{C1}}{dt} + \frac{v_{C1}}{L_1} = 0$$

And now for a little magic. I will **re-scale** time so that

$$\tau = \frac{t}{\sqrt{L_1C_1}}$$

A **non-linear oscillator...**



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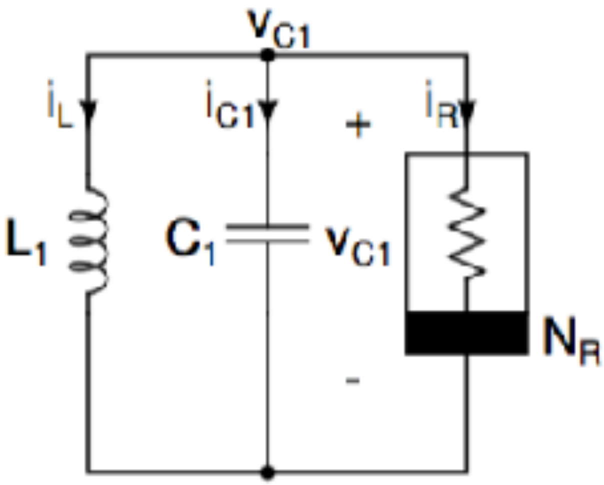
$$C_1 \frac{d^2 v_{C1}}{dt^2} + \left(-1 + v_{C1}^2\right) \frac{1}{R} \frac{dv_{C1}}{dt} + \frac{v_{C1}}{L_1} = 0$$

$$\frac{dv_{C1}}{d\tau} - \epsilon(1 - v_{C1}^2) \frac{dv_{c1}}{d\tau} + v_{c1} = 0$$, where… $\epsilon = \frac{1}{R}\sqrt{\frac{L_1}{C_1}}$

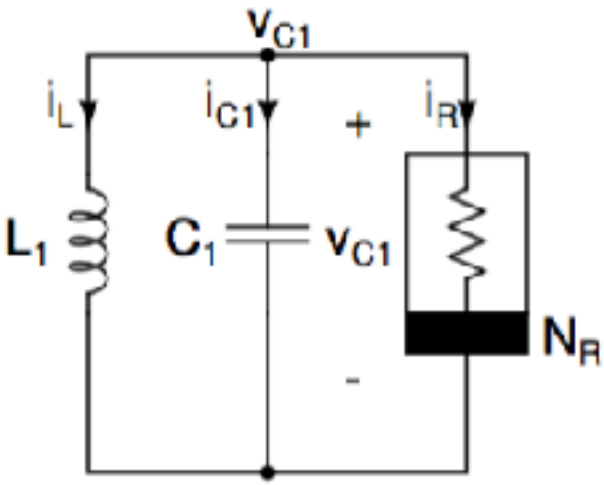
A **non-linear oscillator...**



$$\ddot{v}_{C1} - \epsilon(1 - v_{C1}^2)\dot{v}_{C1} + v_{C1} = 0$$, where... $$\epsilon = \frac{1}{R}\sqrt{\frac{L_1}{C_1}}$$

Remember the basic **van der Pol oscillator** equation?

A **non-linear oscillator...**



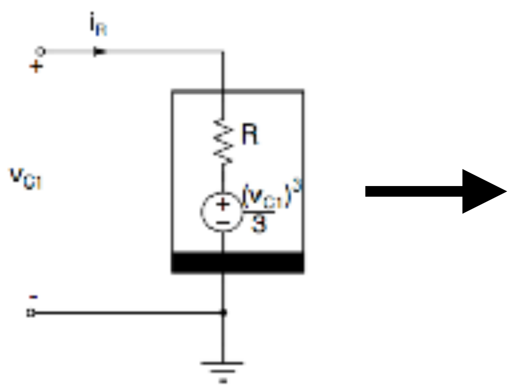
$$\ddot{v}_{C1} - \epsilon(1 - v_{C1}^2)\dot{v}_{C1} + v_{C1} = 0$$, where... $$\epsilon = \frac{1}{R}\sqrt{\frac{L_1}{C_1}}$$

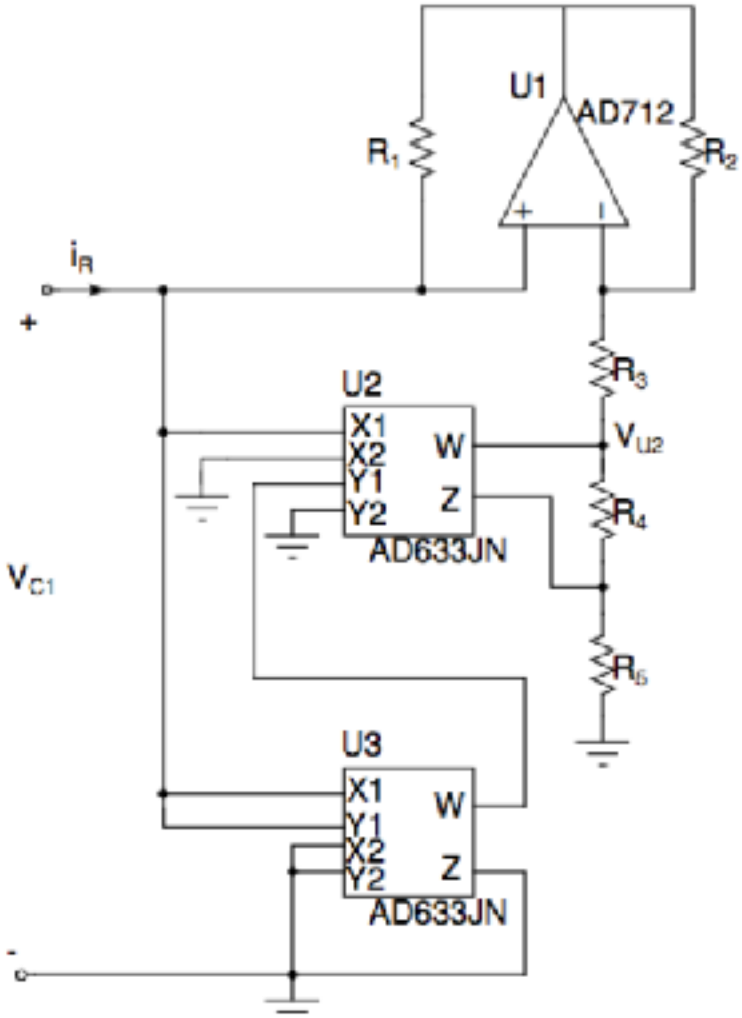
Remember the basic **van der Pol oscillator** equation?

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

A **non-linear oscillator...**

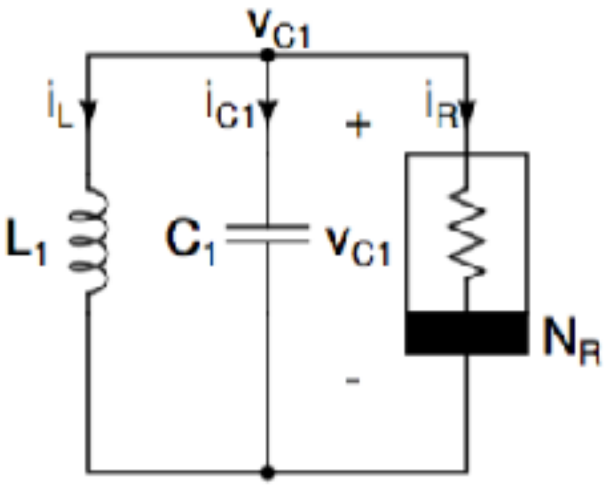
$$i_R(v_{C1}) = (-v_{C1} + \frac{1}{3}v_{C1}^3)\frac{1}{R}$$





…the **actual implementation** of our non-linear resistor element, with appropriate choices of R1-R5 to get a=1, b=1/3, and the effective net resistance to be R

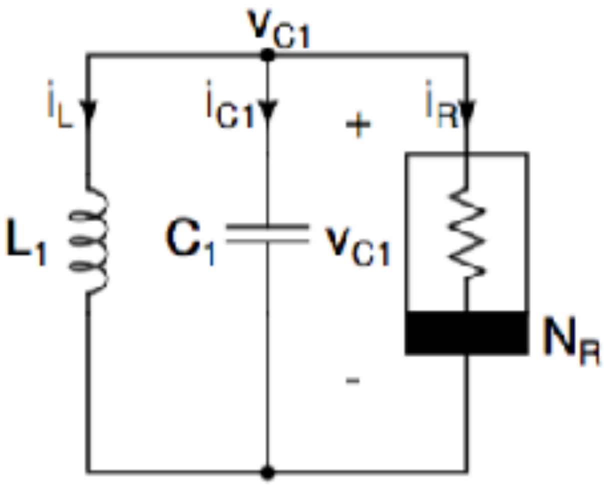
A **non-linear oscillator...**



$$\ddot{v}_{C1} - \epsilon(1 - v_{C1}^2)\dot{v}_{C1} + v_{C1} = 0$$, where... $\epsilon = \frac{1}{R}\sqrt{\frac{L_1}{C_1}}$

now…let's carry out **stability** and **bifurcation** analysis of this system

A **non-linear oscillator...**



$$\ddot{v}_{C1} - \epsilon(1 - v_{C1}^2)\dot{v}_{C1} + v_{C1} = 0$$, where… $$\epsilon = \frac{1}{R}\sqrt{\frac{L_1}{C_1}}$$

$$x = v_{C1}, y = \dot{v}_{C1}$$

$$\dot{x} = y$$
$$\dot{y} = \epsilon \cdot y(1 - x^2) - x$$

again, a **re-writing** of our equation to represent the phase space…and we know the fixed point:

$$(x^*, y^*) = (0, 0).$$

To study the stability of the fixed point, we carry out a **local linearization**…and then look at the flow.

A **general non-linear system…**

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and let's say $f(x^*, y^*) = 0$, $g(x^*, y^*) = 0$.

A **general non-linear system…**

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and let's say $f(x^*, y^*) = 0, \quad g(x^*, y^*) = 0.$

that is, (x*,y*) is a fixed point

A **general non-linear system…**

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and let's say $f(x^*, y^*) = 0, \qquad g(x^*, y^*) = 0.$

Now, we introduce a **disturbance** around the fixed point….

$$u = x - x^*, \qquad v = y - y^*$$

A **general non-linear system…**

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and let's say $f(x^*, y^*) = 0$, $g(x^*, y^*) = 0$.

Now, we introduce a **disturbance** around the fixed point….

$$u = x - x^*, \qquad v = y - y^*$$

To see if the disturbance grows or not, we look at the **derivatives**…

$$\dot{u} = \dot{x} \qquad \text{(since } x^* \text{ is a constant)}$$

$$= f(x^* + u, y^* + v) \qquad \text{(by substitution)}$$

$$= f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) \qquad \text{(Taylor series expansion)}$$

$$= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) \qquad \text{(since } f(x^*, y^*) = 0\text{)}.$$

Similar thing for the **disturbance v**….

A **general non-linear system…**

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and let's say $f(x^*, y^*) = 0, \qquad g(x^*, y^*) = 0.$

Now, we introduce a **disturbance** around the fixed point….

$$u = x - x^*, \qquad v = y - y^*$$

To see if the disturbance grows or not, we look at the **derivatives**…

$$\dot{u} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv)$$

$$\dot{v} = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + O(u^2, v^2, uv).$$

…and in **matrix form,**

A **general non-linear system…**

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and let's say $f(x^*, y^*) = 0, \qquad g(x^*, y^*) = 0.$

Now, we introduce a **disturbance** around the fixed point….

$$u = x - x^*, \qquad v = y - y^*$$

To see if the disturbance grows or not, we look at the **derivatives**…

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \text{quadratic terms.}$$

…and **ignoring the quadratic** and higher order terms, since they are tiny….

A **general non-linear system…**

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and let's say $f(x^*, y^*) = 0, \qquad g(x^*, y^*) = 0.$

Now, we introduce a **disturbance** around the fixed point….

$$u = x - x^*, \qquad v = y - y^*$$

To see if the disturbance grows or not, we look at the **derivatives**…

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

…this is the **locally linearized** form of our general non-linear system…

A **general non-linear system…**

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and let's say $f(x^*, y^*) = 0, \qquad g(x^*, y^*) = 0.$

Now, we introduce a **disturbance** around the fixed point….

$$u = x - x^*, \qquad v = y - y^*$$

To see if the disturbance grows or not, we look at the **derivatives**…

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

this matrix is called the **Jacobian**, and evaluated at the fixed point (x*,y*), tells us about the flow of the system in the local environment…

A **general non-linear system…**

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and let's say $f(x^*, y^*) = 0$, $g(x^*, y^*) = 0$.

Now, we introduce a **disturbance** around the fixed point….

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To see if the disturbance grows or not, we look at the **derivatives**…

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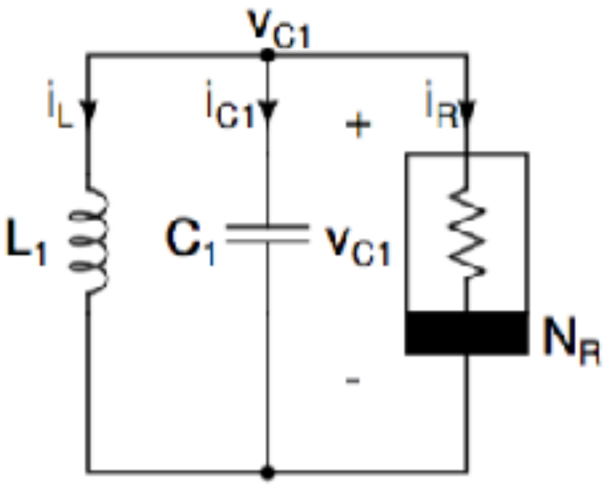
the **Jacobian**. In this form, this matrix is just like the characteristic matrix for a linear system, right? So, we know how to analyze its behavior…

A **non-linear oscillator...**



$$\ddot{v}_{C1} - \epsilon(1 - v_{C1}^2)\dot{v}_{C1} + v_{C1} = 0$$, where… $\epsilon = \frac{1}{R}\sqrt{\frac{L_1}{C_1}}$

$x = v_{C1}, y = \dot{v}_{C1}$

$$\dot{x} = y$$
$$\dot{y} = \epsilon \cdot y(1 - x^2) - x$$

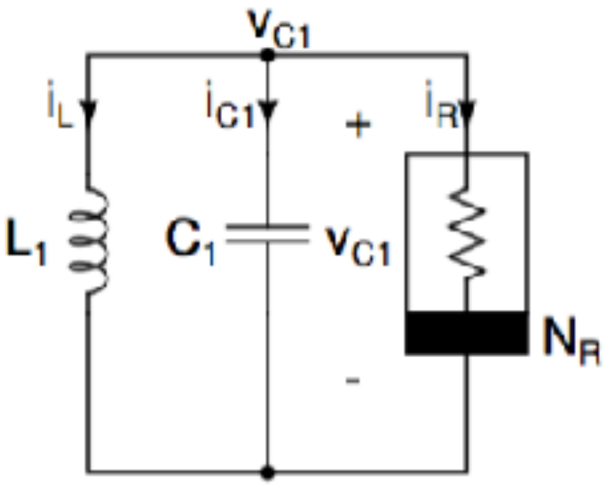
At the fixed point…. $(x^*, y^*) = (0, 0)$.

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix}$$

And what are the **eigenvalues**? Remember that stability is about the sign of the real part of the system eigenvalues…

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

A **non-linear oscillator...**



$$\ddot{v}_{C1} - \epsilon(1 - v_{C1}^2)\dot{v}_{C1} + v_{C1} = 0$$, where… $\epsilon = \frac{1}{R}\sqrt{\frac{L_1}{C_1}}$

$\downarrow$ $x = v_{C1}, y = \dot{v}_{C1}$

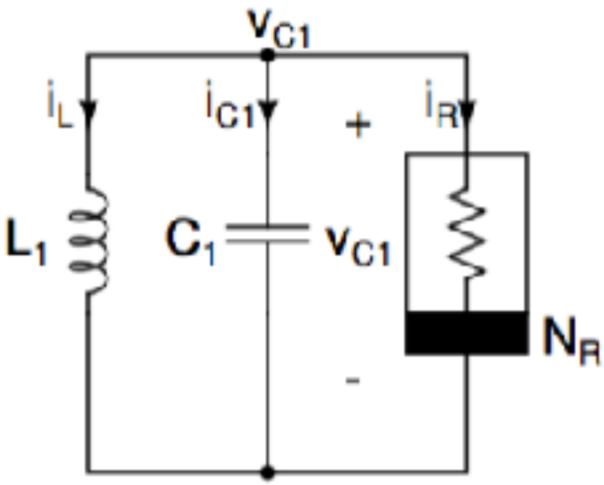
$$\dot{x} = y$$
$$\dot{y} = \epsilon \cdot y(1 - x^2) - x$$

At the fixed point…. $(x^*, y^*) = (0, 0)$.

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix}$$

The **trace of the Jacobian** is epsilon, and so the system is stable for negative values and unstable for positive…

A **non-linear oscillator...**



$$\ddot{v}_{C1} - \epsilon(1 - v_{C1}^2)\dot{v}_{C1} + v_{C1} = 0$$, where… $\epsilon = \frac{1}{R}\sqrt{\frac{L_1}{C_1}}$

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$$\dot{x} = y$$
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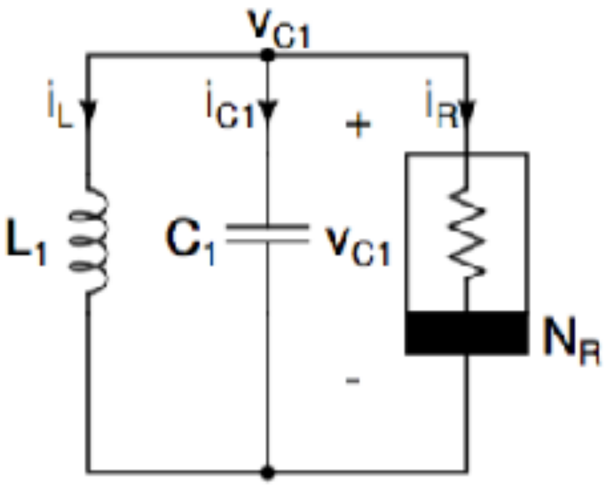
At the fixed point…. $(x^*, y^*) = (0, 0)$.

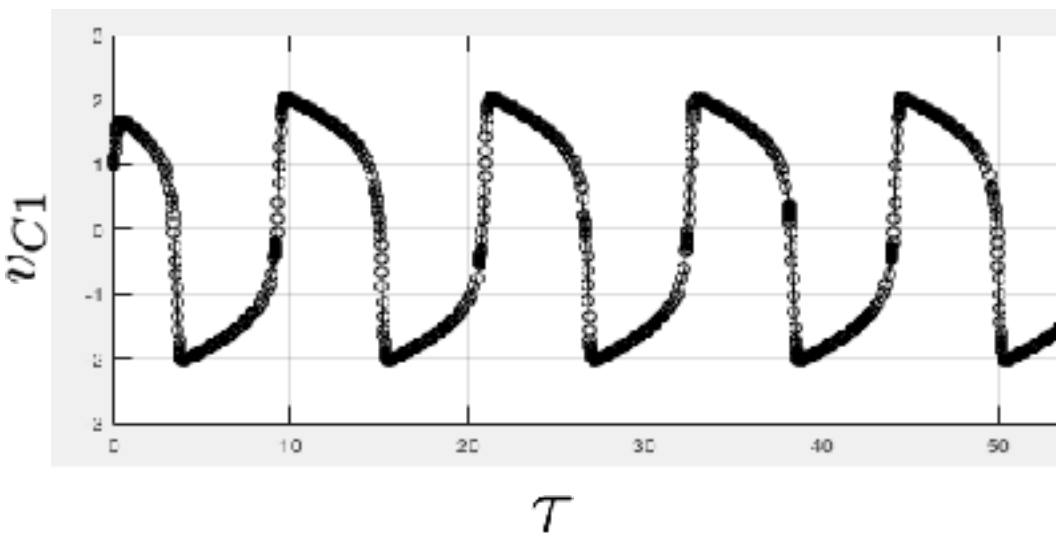
$$J = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix}$$

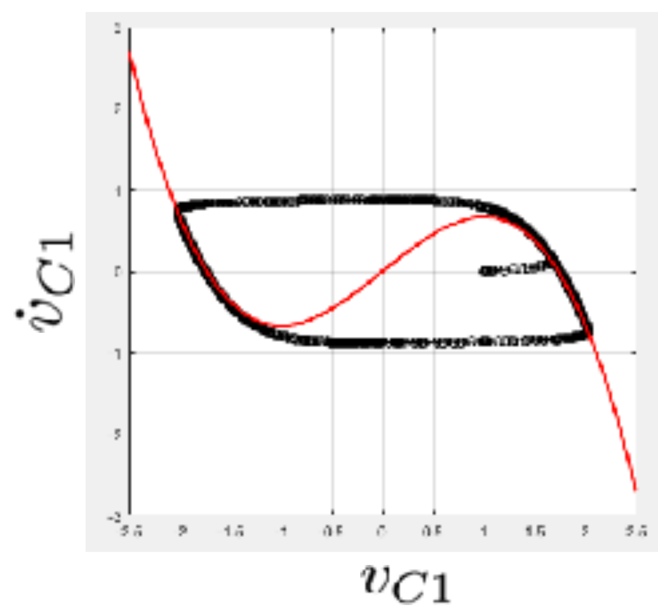
So, for a van der Pol oscillator, the system loses stability at $\epsilon = 0$ the place where the real part of the system eigenvalues go from negative to positive….such an event is called a **Hopf bifurcation**

A **non-linear oscillator...**



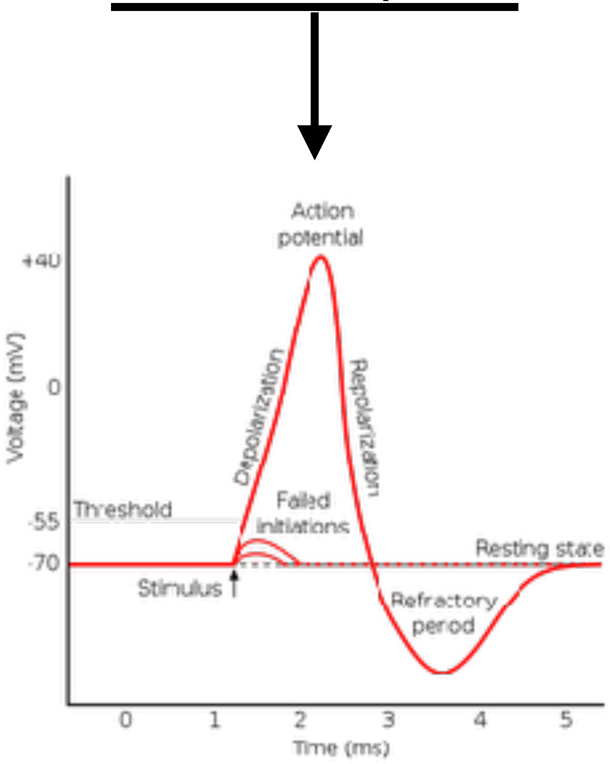
$$\ddot{v}_{C1} - \epsilon(1 - v_{C1}^2)\dot{v}_{C1} + v_{C1} = 0$$, where... $$\epsilon = \frac{1}{R}\sqrt{\frac{L_1}{C_1}}$$



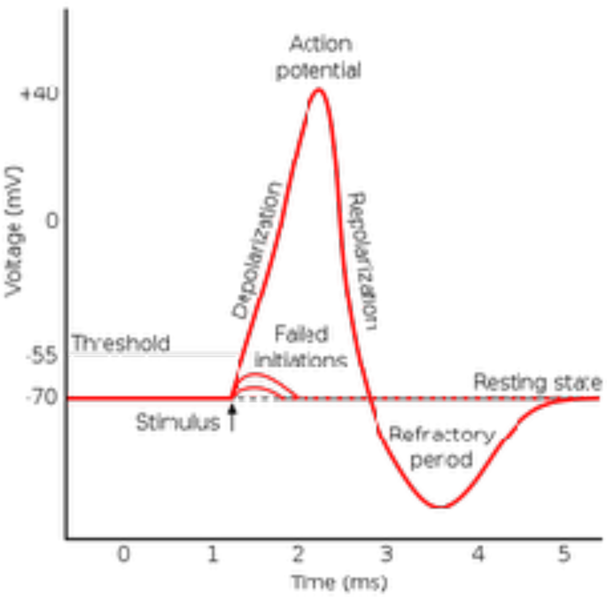


A **non-linear oscillator**...

The **neuronal action potential…**a slight variation on the van der Pol oscillator…



A **non-linear oscillator...**



Fitzhugh-Nagumo (1962)

membrane pot $$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I$$

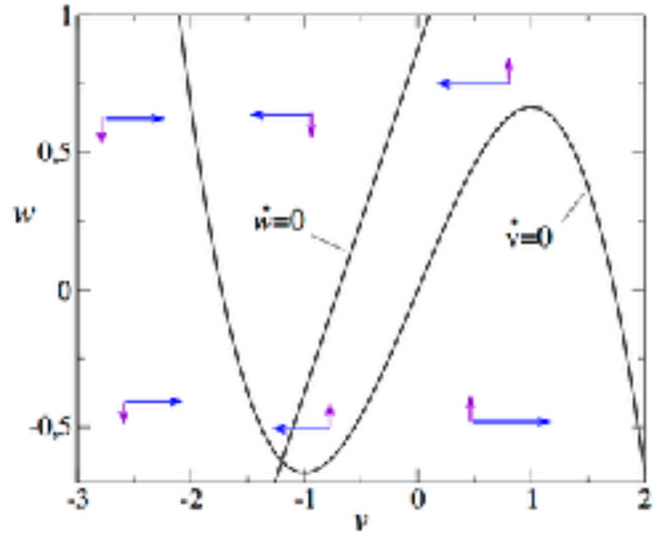
slow K+ flux $$\frac{dw}{dt} = \frac{1}{\tau}(v + a - bw)$$

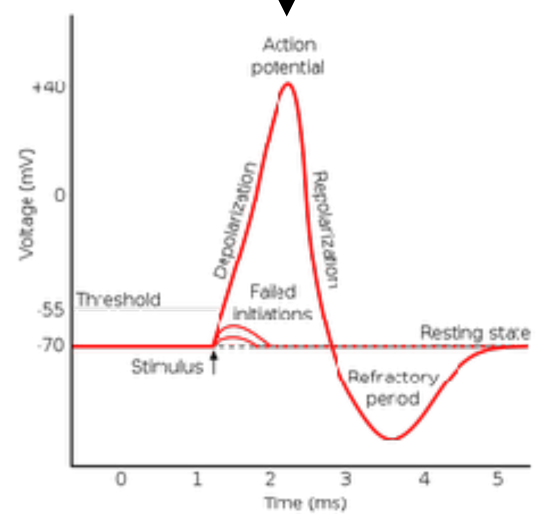
this is essentially the van der Pol oscillator, with **one difference**….

A **non-linear oscillator...**

membrane pot
$$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I$$

slow K+ flux
$$\frac{dw}{dt} = \frac{1}{\tau}(v + a - bw)$$



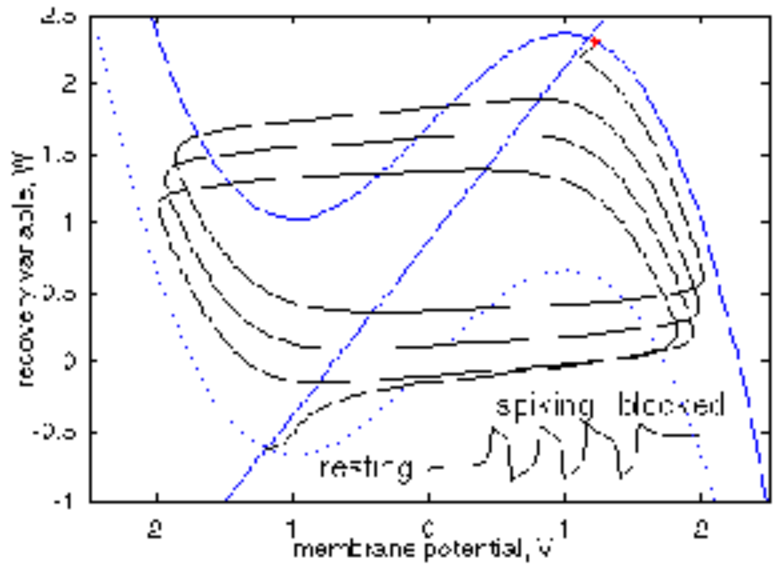


the linear term to the w nullcline provides for **thresholded oscillation**….

A **non-linear oscillator...**

membrane pot
$$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I$$

slow K+ flux
$$\frac{dw}{dt} = \frac{1}{\tau}(v + a - bw)$$



the linear term to the w nullcline provides for **thresholded oscillation**….a stable fixed point destabilized to produce relaxation oscillations. We will look at this more closely next time…

Next, we will further analyze the simple non-linear oscillator systems…

	n = 1	n = 2 or 3	n >> 1	continuum
Linear	exponential growth and decay single step conformational change fluorescence emission pseudo first order kinetics	second order reaction kinetics linear harmonic oscillators simple feedback control sequences of conformational change	electrical circuits molecular dynamics systems of coupled harmonic oscillators equilibrium thermodynamics diffraction, Fourier transforms	Diffusion Wave propagation quantum mechanics viscoelastic systems
Nonlinear	fixed points bifurcations, multi stability irreversible hysteresis overdamped oscillators	anharmomic oscillators relaxation oscillations predator-prey models van der Pol systems Chaotic systems	systems of non-linear oscillators non-equilibrium thermodynamics protein structure/ function neural networks the cell ecosystems	Nonlinear wave propagation Reaction-diffusion in dissipative systems Turbulent/chaotic flows

adapted from S. Strogatz