Lecture 98  Non-linear Dynamical Systems - Part 1

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Qualitative analysis and principles of non-linear dynamical systems

Joseph-Louis LaGrange 1736 - 1813

Henri Poincare 1854 - 1912

Edward Lorenz 1917-2008

Mitchell Feigenbaum 1944 -

Robert May 1936 -

Albert Libchaber 1934 -
So, today we explore the truly astounding emergent complexity inherent in even simple **non-linear dynamical systems**.

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adapted from S. Strogatz
So, today:

(1) A reminder of the **reducibility**, **simplicity**, and **predictability** of linear systems….we needed to understand what is not “complex” first!

(2) A case study of three small non-linear dynamical systems that exhibit remarkable **emergent** and **non-obvious** behaviors that linear systems cannot do. All relevant for biology…
We begin with a reminder of **linear systems**...

---

*Linearity implies a principle called superposition:* If $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the response to $x_2(t)$, then:

1. $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$  
   \[\text{[additivity]}\]

2. $a \cdot x_1(t) \rightarrow a \cdot y_1(t)$  
   \[\text{[scaling or homogeneity]}\]
We begin with a reminder of **linear systems**...

**Linearity implies a principle called superposition**: If \( y_1(t) \) is the output of a system to input \( x_1(t) \) and \( y_2(t) \) is the response to \( x_2(t) \), then:

1. \[ x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t) \quad \text{[additivity]} \]
2. \[ a \cdot x(t) \rightarrow a \cdot y(t) \quad \text{[scaling or homogeneity]} \]

So... if input is

\[ x(t) = \sum_k a_k x_k(t) = a_1 x_1(t) + a_2 x_2(t) + \ldots \]

output will be:

\[ y(t) = \sum_k a_k y_k(t) = a_1 y_1(t) + a_2 y_2(t) + \ldots \]

This is superposition... the output is a weighted sum of responses to independent inputs.
...a second-order biochemical reaction, for example
The general solution to second-order linear system...

\[ \begin{align*}
\dot{x} &= ax + by \\
\dot{y} &= cx + dy \\
\end{align*} \]

\[ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

\[ \dot{x} = Ax \]

given \[ x_0 \] ...a vector of initial conditions

\[ x(t) = e^{At}x_0 \]

...where \( A \) is the characteristic matrix. It’s properties control all behaviors of the system.
Properties of the characteristic matrix:

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

\[ \tau = \text{trace}(A) = a + d, \]
\[ \Delta = \det(A) = ad - bc. \]

...the \textbf{trace} and \textbf{determinant} of the matrix
Properties of the characteristic matrix…

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
\[ \tau = \text{trace}(A) = a + d, \]
\[ \Delta = \det(A) = ad - bc. \]

\[ \lambda_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\Delta} \right) \]

…the eigenvalues of \( A \) are completely determined by the trace and determinant…
System behaviors: the second order linear case

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \lambda_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\Delta} \right) \]

…the zoo of all possible behaviors for a linear, second-order system
System behaviors: the second order linear case

\[ \lambda_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\Delta} \right) \]

...stable nodes
System behaviors: the second order linear case

\[ \lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta}) \]

...saddle nodes
System behaviors: the second order linear case

\[ \lambda_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\Delta} \right) \]

...and spiral nodes when eigenvalues are complex numbers
System behaviors: the second order linear case

\[ \lambda_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\Delta} \right) \]

...both spiral nodes when eigenvalues are complex numbers
Seeing behaviors: the linear harmonic oscillator

Often, it is hard to get analytic solutions. We need a way of “seeing” system behavior....
Seeing behaviors: the linear harmonic oscillator

Now, the equation of motion is:

\[ F = ma \quad \text{or...} \]

\[ -kx = m \ddot{x} \]

Remember that for a Hooke spring,

\[ F = -kx \quad \text{and} \]

\[ \dot{x} = \frac{dx}{dt} = v \]

\[ \ddot{x} = \frac{d^2x}{dt^2} = a \]
We can **re-write** this equation….
Seeing behaviors: the linear harmonic oscillator

\[ m \ddot{x} + k x = 0 \]

\[ \dot{v} = v \]
\[ \dot{v} = -\frac{k}{m} x \]

The first equation is just the definition of velocity. The second equation is \( m \ddot{x} + k x = 0 \) re-written in terms of \( v \).
Seeing behaviors: the linear harmonic oscillator

To simplify, we define $\omega^2 = \frac{k}{m}$. So...

\[ \begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2 x
\end{align*} \]
Seeing behaviors: the linear harmonic oscillator

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2 x
\end{align*}
\]

\[
\begin{bmatrix}
\dot{x} \\
\dot{v}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-\omega^2 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
v
\end{bmatrix}
\]

We know how to analyze the behavior, right?
Seeing behaviors: the linear harmonic oscillator

\[ \dot{x} = v \\
\dot{v} = -\omega^2 x \]

\[ \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \]

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

\[ \lambda_{1,2} = \frac{1}{2} (\tau \pm \sqrt{\tau^2 - 4\Delta}) \]

Diagram showing the behavior of the linear harmonic oscillator with regions for stable nodes, unstable nodes, saddle points, centers, and non-isolated fixed points.
Seeing behaviors: the linear harmonic oscillator

\[ \begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2 x
\end{align*} \]

\[
\begin{pmatrix}
\dot{x} \\
\dot{v}
\end{pmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega^2 & 0
\end{bmatrix}
\begin{pmatrix}
x \\
v
\end{pmatrix}
\]

\[ \text{Trace} = 0 \]
\[ \text{Det} = \omega^2 \]

so...

\[
\lambda_{1,2} = \frac{1}{2} \left[ 0 \pm \sqrt{0 - 4\omega^2} \right]
\]

\[ = \pm i\omega \]
Seeing behaviors: the linear harmonic oscillator

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2 x
\end{align*}
\]

\[
\begin{bmatrix}
\dot{x} \\
\dot{v}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega^2 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
v
\end{bmatrix}
\]

\[
\lambda_{\pm} = \frac{1}{2} \left[ \omega \pm \sqrt{\omega^2 - 4\Delta} \right]
\]

Diagram showing the stability regions for different values of \( \tau \) and \( \Delta \).
Seeing behaviors: the linear harmonic oscillator

\[ x \quad \begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2 x \end{aligned} \]

Thus, for every \((x,v)\), this system of equations assigns a vector \((\dot{x}, \dot{v}) = (v, -\omega^2 x)\). This makes a vector field...

....for this system, \((x,v)\) represents a 2D “phase space” in which we can see the behavior of the system intuitively.
Seeing behaviors: the linear harmonic oscillator

plot the system nullclines...
Seeing behaviors: the linear harmonic oscillator

\[ \begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2 x
\end{align*} \]

\((x, v) = (0, -\omega^2 x)\)

\((\dot{x}, \dot{v}) = (v, 0)\)
Seeing behaviors: the linear harmonic oscillator

This is called a “phase portrait”... a way of seeing system dynamics.
A summary...

Linear systems are:

(1) **decomposable**, such that high-order systems are combinations of first-order systems. This is the concept that the behavior of the whole is predictable from knowledge of the behavior of the underlying parts.

(2) **understandable**; their behavior can be mapped through a study of their so-called eigenfunctions. This is the concept that one can “understand” the properties of linear systems by sketching the behavior of these eigenfunctions.

(3) **simple**; these systems show single fixed points…whether stable or unstable.
A non-linear oscillator...

Here is the non-linearity… with \( \mu \) controlling the degree of non-linearity.
A non-linear oscillator...

\[ \ddot{x} + \mu(x^2 - 1) \dot{x} + x = 0 \]  
(The van der Pol oscillator)

For \( \mu >> 1 \), this is the strongly non-linear limit. There is a position-dependent damping term \( \mu(x^2 - 1)x \). This acts like positive damping for \( |x| > 1 \) to cause oscillations to decay, but acts like negative damping for \( |x| < 1 \) to build oscillations up.

It causes so-called "relaxation oscillations"...
A non-linear oscillator...

\[ \ddot{x} + \mu (x^2-1) \dot{x} - x = 0 \]

Let's write the equation in the usual \( \dot{x} \), \( y \) way to make a phase plane portrait...

A little re-definition of variables... note that

\[ \ddot{x} + \mu (x^2-1) \dot{x} + x = 0 \]  \hspace{1cm} (1)

\[ \ddot{x} + \mu (x^2-1) \dot{x} = \frac{1}{\delta \epsilon} \left( \dot{x} + \mu \left[ \frac{1}{2} x^3 - x \right] \right) \]  \hspace{1cm} (2)

\[ F(x) = \frac{1}{3} x^3 - x \]  \hspace{1cm} (3)

\[ \omega = \dot{x} + \mu F(x) \]  \hspace{1cm} (4)

---

So...

\[ \omega = \ddot{x} + \mu (x^2-1) \dot{x} = -x \]  \hspace{1cm} using (1), (2), (3)

Re-writing the equations in a more intuitive way....
A non-linear oscillator...

\[ \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \]

we can now rewrite the van der Pol eqn (1) as...

\[ \dot{x} = \omega - \mu F(x) \]
\[ \dot{\omega} = -x \]
A non-linear oscillator...

\[ \ddot{x} + \mu(x^2 - 1) \dot{x} + x = 0 \]

We can now re-write the van der Pol eqn (1) as...

\[ \dot{x} = \omega - \mu \phi(x) \]
\[ \dot{\omega} = -x \]

One other convenience... set \( y = \frac{\omega}{\mu} \). Then...

\[ \dot{x} = \mu[y - F(x)] \]
\[ \dot{y} = -\frac{1}{\mu} x \]
A non-linear oscillator...

\[ \ddot{x} + \mu(x^2 - 1) \dot{x} + x = 0 \]

We can now re-write the van der Pol eqn (1) as:

\[ \begin{align*}
\dot{x} &= \omega - \mu F(x) \\
\dot{\omega} &= -x
\end{align*} \]

For other convenience... set \( y = \frac{x}{\mu} \). Then...

\[ \begin{align*}
\dot{x} &= \mu [y - F(x)] \\
\dot{y} &= -\frac{1}{\mu} x
\end{align*} \]

Now, to see behavior of our system, we sketch the so-called "nullclines" of the system... the equations corresponding to \( \dot{x} = 0 \) and \( \dot{y} = 0 \).
The van der Pol oscillator. Therefore,

\[ \ddot{x} + \mu (x^2 - 1) \dot{x} + x = 0 \]

The system is described by:

\[ \ddot{x} = \mu [y - F(x)] \]
\[ \dot{y} = -\frac{1}{\mu} x \]

Nullclines are:

- \( y = 0 \)
- \( y = F(x) \) with \( F(x) = \frac{1}{3} x^3 - x \) [a cubic function]
The van der Pol oscillator....

\[ \ddot{x} + \mu(x^2 - 1) \dot{x} + x = 0 \]

\[ \dot{x} = \mu [y - F(x)] \]
\[ \dot{y} = -\frac{1}{\mu} x \]

Nullclines are...

\[ y = 0 \quad \text{and} \quad y = F(x) \]
\[ y = \frac{1}{3} x^3 - x \quad \text{[a cubic function]} \]

Plots...
The van der Pol oscillator...

\[ \ddot{x} + \mu (x^2 - 1) \dot{x} + x = 0 \]

\[ \dot{x} = \mu [y - F(x)] \]
\[ \dot{y} = -\frac{1}{\mu} x \]

**Nullclines are...**

\[ y = 0 \] , and...
\[ y = F(x) = \frac{1}{3} x^3 - x \quad \text{[a cubic function]} \]

Plots...

1. Take \((x_0, y_0) \) far from nullclines. \( \Rightarrow \) say \( y = F(x) \sim O(\mu) \)
2. \( \mu \gg 1 \)
3. \( |\dot{x}| \sim O(\mu^0) \) \( \gg 1 \)
4. \( |\dot{y}| \sim O(\mu^{-1}) \ll 1 \)

So...the flows are all horizontal and fast!

Now if \( y = F(x) \sim O(\mu^{-2}) \) ...that is, close to cubic nullcline,
\[ |\dot{x}| \sim O(\mu^{-2}) \]
\[ |\dot{y}| \sim O(\mu^{-1}) \]

...flows equal in both directions ...and slow!
The van der Pol oscillator...

\[ \ddot{x} + \mu(x^2 - 1) \dot{x} + x = 0 \]

\[ \dot{x} = \mu [y - F(x)] \]
\[ \dot{y} = -\frac{1}{\mu} x \]

Nullclines are...

- \( y = 0 \), and...

- \( y = F(x) \)
  \[ y = \frac{1}{3} x^3 - x \] [a cubic function]

Plots...

1. Take \((x_0, y_0)\) far from nullclines. # say \( y = F(x) \sim O(1) \)
2. If \( y \sim O(\mu) \gg 1 \) (because \( \mu >> 1 \)),
   \[ |\dot{y}| \sim O(\mu^{-1}) \ll 1 \]

So... the flow's all horizontal and fast!

(Now if \( y = F(x) \sim O(\mu^{-2}) \) ... that is, close to cubic nullline,
\[ |\dot{x}| \sim O(\mu^{-1}) \]
\[ |\dot{y}| \sim O(\mu^{-2}) \]

...flow equal in both directions ... and slow!

Thus plotting \( x(t) \):

\[ x \sim O(\mu^{-1}) \]
\[ \langle \sim O(\mu) \rangle \]
We will study the behavior of these systems in more detail next time, but as a preview, the basic model for the **neuronal action potential** is only a slight variation on the van der Pol oscillator…

A non-linear oscillator…
We will study the behavior of these systems in more detail next time, but as a preview, the basic model for the **neuronal action potential** is only a slight variation on the van der Pol oscillator...

A non-linear oscillator...

---

A regenerative, **non-linear activation** with a sharp **threshold**, an all-or-nothing character, and a **refractory period** afterwards...
A non-linear oscillator...

Hodgkin-Huxley (1952)

\[
\begin{align*}
\frac{dV}{dt} &= C_Na m^3 h (E_{Na} - V) + C_K n^4 (E_K - V) + C_{leak} (V_{rest} - V) + I_{inj(t)} \\
\frac{dm}{dt} &= \alpha_m(V) (1 - m) - \beta_m(V) m \\
\frac{dh}{dt} &= \alpha_h(V) (1 - h) - \beta_h(V) h \\
\frac{dn}{dt} &= \alpha_n(V) (1 - n) - \beta_n(V) n
\end{align*}
\]

membrane pot
fast Na+ flux
slow Na+ flux
slow K+ flux

but, Fitzhugh and Nagumo simplified this 4D set of equations....
A non-linear oscillator...

Fitzhugh-Nagumo (1962)

\[
\begin{align*}
\frac{dv}{dt} &= v - \frac{v^3}{3} - w + I \\
\frac{dw}{dt} &= \frac{1}{\tau}(v + a - bw)
\end{align*}
\]

membrane pot

slow K+ flux

this is essentially the van der Pol oscillator, with one difference....
A non-linear oscillator...

membrane pot

\[ \frac{dv}{dt} = v - \frac{v^3}{3} - w + I \]

\[ \frac{dw}{dt} = \frac{1}{\tau} (v + a - bw) \]

the linear term to the \( w \) nullcline provides for **thresholded oscillation**...you will see next time
A seemingly innocuous thing… the so-called logistic equation
But led to principles have **broad application** in both basic and applied science….and art, social science, and the popular media.
A seemingly innocuous thing….the so-called logistic equation
An iterative map gives the current value of a system as a function of its previous value...

\[ f(s) = g(s(1-s)) \]

Let's see... using a graphical method called an iterative map:

This plots the value of a function against its previous value...

\[ f(s) = F\{ f(s-1) \} \]
\[ f(s) = F\{ f(s-1) \}\]  

Example:

\[ f(s+1) = f(s) + c \quad ; \quad \text{Say} \quad f(0) = s_0 \]
\[ f(0) = s_0 \]
\[ f(1) = s_0 + c \]
\[ f(2) = s_0 + c + c = s_0 + 2c \]
The equation for constant velocity motion….
Note that these maps are discrete time mappings! Also, we are now discussing just 1-D maps. Why 1D? Only one variable we are following:

\[ f(s) = F\{f(s-1)\} \]
Fixed points: a more formal treatment

**Fixed points:**

Say a certain value $f(x)$ satisfies the rule that

\[ f(x) = x^n \]

Then this value $f(x)$ is called a **fixed point**, because the orbit stays at $x^n$ for all future values $f(x)$.

**Stability of the fixed point:**

To determine stability, the idea is to cause a small perturbation and ask whether the orbit is attracted back to $x^n$ or is repelled.

[Diagrams of stable and unstable steady states]
Fixed points...a more formal treatment

so consider...

\[ x_n = x^* + \eta_n \]

- perturbation

\[ x_{n+1} = x^* + \eta_{n+1} = f(x^* + \eta_n) \]

\[ = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2) \]

What is this?
Fixed points….a more formal treatment

So consider...

\[ x_n = x^2 + \eta_n \]

\[ x_{n+1} = x_n + \eta_{n+1} = f(x_n + \eta_n) \]

\[ = f(x^n) + f'(x^n) \eta_n + O(\eta_n^2) \]

What is this? Well, the higher order stuff, which we will conveniently ignore….
Fixed points...a more formal treatment

So consider...

\[ x_n = x^2 + \eta_n \]

\[ x_{n+1} = x^2 + \eta_{n+1} = f(x^2 + \eta_n) = f(x^2) + f'(x^2)\eta_n + O(\eta_n^2) \]

But \( f(x^2) = x^4 \), so...

\[ \eta_{n+1} = f'(x^4)\eta_n \]

(all the "multiplier" of the perturbation \( \lambda = f'(x^4) \). Then

\[ \eta_1 = \lambda \eta_0 \]
\[ \eta_2 = \lambda^2 \eta_0 \]
\[ \eta_3 = \lambda^3 \eta_0 \]
\[ \vdots \]
\[ \eta_n = \lambda^n \eta_0 \]
Fixed points... a more formal treatment

So consider...

\[ x_n = x^n + \eta_n \]

(All the "multiplier" of the perturbation \( \lambda = f'(x^*) \). Then

\[ \eta_1 = \lambda \eta_0 \]
\[ \eta_2 = \lambda^2 \eta_0 \]
\[ \vdots \]
\[ \eta_n = \lambda^n \eta_0 \]

So... if \(|\lambda| < 1\), then \( \eta_n \to 0 \) as \( n \to \infty \) [stable fixed pt]

if \(|\lambda| > 1\), then \( \eta_n \to \infty \) as \( n \to \infty \) [unstable]
Fixed points… a more formal treatment

Example: $x_{n+1} = x_n^2$

Fixed points?
Fixed points... a more formal treatment

Example: $x_{n+1} = x_n^2$

Fixed points? 0 or 1.

Stability?
Fixed points,...a more formal treatment

Example: \( x_{n+1} = x_n^2 \)

Fixed points? 0 or 1.

Stability?

\( \lambda = f'(x^n) = 2x^n \)
Fixed points … a more formal treatment

Example: \( x_{n+1} = x_n^2 \)

Fixed points? 0 or 1.
Stability?
\( \lambda = f'(x^n) = 2x^n \)

So… the fixed point at zero is **stable**, and the one at 1 is **not**.

We will do a similar analysis for the logistic equation soon.
Iterative Maps

\[ x_{n+1} = x_n + c \], where \( c \) is a constant. Say \( x_0 = 3.5 \ldots \) \[ c = 10 \]
Iterative Maps

\[ x_{n+1} = x_n + c \]

where \( c \) is a constant. Say \( x_0 = 3.5 \), \( c = 10 \).

Just a way of plotting the “orbit”…or the behavior of the equation.
Iterative Maps

In this plot, what would a **fixed point** be? Are there any?
Iterative Maps

This is a really boring equation....
Iterative Maps

This is a **really boring** equation….but it does have a fixed point! What about stability of the fixed point?
Iterative Maps

\( x_{n+1} = C \), say \( x_0 = 3.5 \), \( C = 10 \)

Well, \( x = 10 \) is a fixed point. Stable?

The places where a function crosses the identity trace are the fixed points.
Iterative Maps

So... now for the so-called logistic equation

$$x_{n+1} = rx_n(1-x_n)$$

Ok.....but where is the **identity trace** relative to the curve?
So... now for the so-called logistic equation

\[ x_{n+1} = r x_n (1 - x_n) \]

Ok.....but where is the identity trace relative to the curve? Well....it depends on \( r \)...
Iterative Maps

\[ y(n + 1) = ry(n)(1 - y(n)) \]

Ok…..now, where are the **fixed points** and what about **stability**?
Analysis of the logistic map.

\[ x_{n+1} = r x_n (1 - x_n) \]

\[ 0 \leq x_n \leq 1 \]

\[ 0 \leq r \leq 4 \quad \text{... the intensity range.} \]
Analysis of the logistic map.

\[ x_{n+1} = r x_n (1 - x_n) \quad 0 \leq x_n \leq 1 \]
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(1) Find fixed points.

\[ x^* = f(x^*) = r x^* (1 - x^*) \quad \text{where is this true?} \]
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\[ x^* = f(x^*) = r x^* (1 - x^*) \quad \text{where is this true?} \]

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\[ x^* = 1 - \frac{1}{r} \]
Analysis of the logistic map.

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(2) Stability

well... \[ \lambda = f'(x^*) = r - 2rx^* \]

For \( x^* = 0 \), the origin is stable for \( r < 1 \) (since \( f'(x^*) = r \))
unstable for \( r > 1 \)
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For \( x^* = 1 - \frac{1}{r} \), \( f'(x^*) = 2 - r \). So...

stable for \( 1 < r < 3 \)
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What happens for $r > 3$?
What happens for $r > 3$?
Let’s look at the dynamics of this equation. We will start with \( y(0) = 0.9 \), and consider 100 iterations at various values of \( r \). Remember that \( r \) is basically the feedback strength in our small positive feedback reaction scheme....
The spectacular consequences of a small bit of non-linearity....

\[ y(n + 1) = ry(n)(1 - y(n)) \]

\[ y(0) = 0.9, \quad r = 0.9 \]

well... \[ \lambda = f'(x^n) = r - 2rx^n \]

for \( x^n = 0 \), the origin is stable for \( r < 1 \) (since \( f'(x^n) = r \))

unstable for \( r > 1 \)
The spectacular consequences of a small bit of non-linearity....

\[ y(n + 1) = ry(n)(1 - y(n)) \]

\[ y(0) = 0.9, \quad r = 2.0 \]

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\[ y(n + 1) = ry(n)(1 - y(n)) \]

\[ y(0) = 0.9, \quad r = 2.9 \]

For \( x^* = 1 - \frac{1}{r} \), \( f'(x^*) = 2 - r \). So...

stable for \( 1 < r < 3 \)
unstable for \( r > 3 \)
The spectacular consequences of a small bit of non-linearity....

\[ y(n + 1) = ry(n)(1 - y(n)) \]

\[ y(0) = 0.9, \ r = 3.1 \]

So, this is called a 2-cycle. That is, both fixed points have lost stability, and we have a system that is said to have bifurcated.
\[ y(n + 1) = r y(n)(1 - y(n)) \]

\[ y(0) = 0.9, \ r = 3.45 \]

Now…what happened? Even the two-cycle has lost stability! What do we have now?
The spectacular consequences of a small bit of non-linearity....

\[ y(n + 1) = r y(n)(1 - y(n)) \]

\[ y(0) = 0.9, \ r = 3.53 \]

So...a **4-cycle**. The system is said to have bifurcated again, or period doubling

The spectacular consequences of a small bit of non-linearity....

\[ y(n + 1) = r y(n)(1 - y(n)) \]

\[ y(0) = 0.9, \ r = 3.56 \]

And…a 8-cycle. Do you notice that the intervals over which our system bifurcates is getting smaller and smaller?
The spectacular consequences of a small bit of non-linearity....

\[ y(n + 1) = r y(n)(1 - y(n)) \]

\[ y(0) = 0.9, \ r = 3.6 \]

And then we come to this....a regime of so-called deterministic chaos.
The spectacular consequences of a small bit of non-linearity....

\[ y(n + 1) = r y(n)(1 - y(n)) \]

\[ y(0) = 0.9, \ r = 3.6 \]

And then we come to this....a regime of so-called **deterministic chaos**. This is characterized by two things: (1) a large number of seemingly constantly changing states, and (2) extreme sensitivity to initial conditions.
Deterministic chaos: sensitivity to initial conditions…. (the “butterfly effect”)

\[ y(n+1) = \frac{1}{4} (1 + \sqrt{1 - 4y(n)^2}) \]

\[ y(0) = 0.9, \quad r = 3.6 \]

\[ y(0) = 0.8999, \quad r = 3.6 \]

\[ y(0) = 0.9001, \quad r = 3.6 \]
Deterministic chaos: the number of states...

\[ y(n + 1) = r y(n)(1 - y(n)) \]

So...what does happen for \( r > 3 \)?
Deterministic chaos: the number of states....

\[ y(n + 1) = r y(n)(1 - y(n)) \]

The famous diagram of period doublings....
Deterministic chaos: the number of states….

\[ y(n + 1) = r y(n)(1 - y(n)) \]

The notions of self-similarity and scale invariance….
Deterministic chaos: the number of states….

\[ y(n + 1) = r y(n) (1 - y(n)) \]

The famous diagram of period doublings… can we mathematically understand every doubling point and the entrance into the regime of chaos?
Next, we will further analyze the simple non-linear oscillator systems...

<table>
<thead>
<tr>
<th>Linear</th>
<th>n = 1</th>
<th>n = 2 or 3</th>
<th>n &gt;&gt; 1</th>
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<tbody>
<tr>
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<td>Diffusion</td>
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<td></td>
<td>fixed points</td>
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<td>diffraction, Fourier transforms</td>
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</table>

| Nonlinear      | anharmonic oscillators                     | systems of non-linear oscillators         | Nonlinear wave propagation                  |
|                | relaxation oscillations                    | non-equilibrium thermodynamics            | Reaction-diffusion in dissipative systems   |
|                | predator-prey models                       | protein structure/function                | Turbulent/chaotic flows                     |
|                | van der Pol systems                        | neural networks                           |                                               |
|                | Chaotic systems                            | the cell                                  |                                               |
|                |                                           | ecosystems                                |                                               |

adapted from S. Strogatz