Lecture 98 Non-linear Dynamical Systems - Part 1

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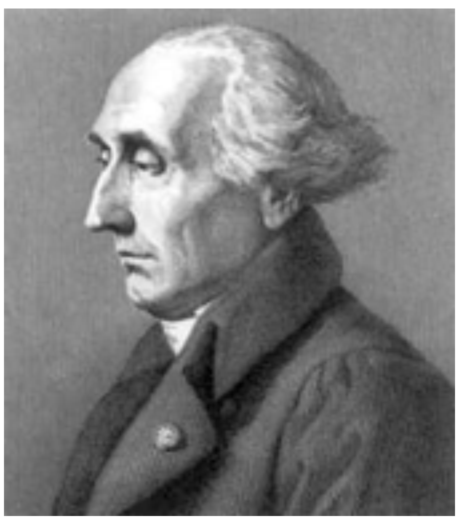
Qualitative analysis and principles of **non-linear** dynamical systems



Joseph-Louis LaGrange
1736 - 1813

Henri Poincare
1854 - 1912

Edward Lorenz
1917-2008

Mitchell Feigenbaum
1944 -

Robert May
1936 -

Albert Libchaber
1934 -

So, today we explore the truly astounding emergent complexity inherent in even simple **non-linear dynamical systems**.

	n = 1	n = 2 or 3	n >> 1	continuum
Linear	exponential growth and decay single step conformational change fluorescence emission pseudo first order kinetics	second order reaction kinetics linear harmonic oscillators simple feedback control sequences of conformational change	electrical circuits molecular dynamics systems of coupled harmonic oscillators equilibrium thermodynamics diffraction, Fourier transforms	Diffusion Wave propagation quantum mechanics viscoelastic systems
Nonlinear	fixed points bifurcations, multi stability irreversible hysteresis overdamped oscillators	anharmonic oscillators relaxation oscillations predator-prey models van der Pol systems Chaotic systems	systems of non-linear oscillators non-equilibrium thermodynamics protein structure/ function neural networks the cell ecosystems	Nonlinear wave propagation Reaction-diffusion in dissipative systems Turbulent/chaotic flows

adapted from S. Strogatz

So, today:

(1) A reminder of the **reducibility**, **simplicity**, and **predictability** of linear systems….we needed to understand what is not "complex" first!

(2) A case study of three small non-linear dynamical systems that exhibit remarkable **emergent** and **non-obvious** behaviors that linear systems cannot do. All relevant for biology…

We begin with a reminder of **linear systems**...

Linearity implies a principle called **superposition**: If $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the response to $x_2(t)$, then:

① $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$ [additivity]

② $a \cdot x_1(t) \longrightarrow a \cdot y_1(t)$ [scaling or homogeneity]

We begin with a reminder of **linear systems**...

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So... if input is

$$x(t) = \sum_k a_k x_k(t) = a_1 x_1(t) + a_2 x_2(t) + \cdots$$

output will be:

$$y(t) = \sum_k a_k y_k(t) = a_1 y_1(t) + a_2 y_2(t) + \cdots$$

This is superposition ... the output is a weighted sum of responses to independent inputs.

$$X \underset{k}{\overset{k}{\rightleftharpoons}} Y$$

$$\dot{x} = -kx + ky$$
$$\dot{y} = kx - ky$$

or….

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -k & k \\ k & -k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

…a **second-order biochemical reaction**, for example

The **general solution** to second-order linear system…

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

$$\rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\dot{\mathbf{x}} = A\mathbf{x}$$ given $\mathbf{x_0}$ …a vector of initial conditions

$$\downarrow$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x_0}$$

….where **A** is the characteristic matrix. It's properties control all behaviors of the system

Properties of the characteristic matrix…

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{aligned} \tau &= \mathrm{trace}(A) = a + d\,, \\ \Delta &= \det(A) = ad - bc\,. \end{aligned}$$

…the **trace** and **determinant** of the matrix

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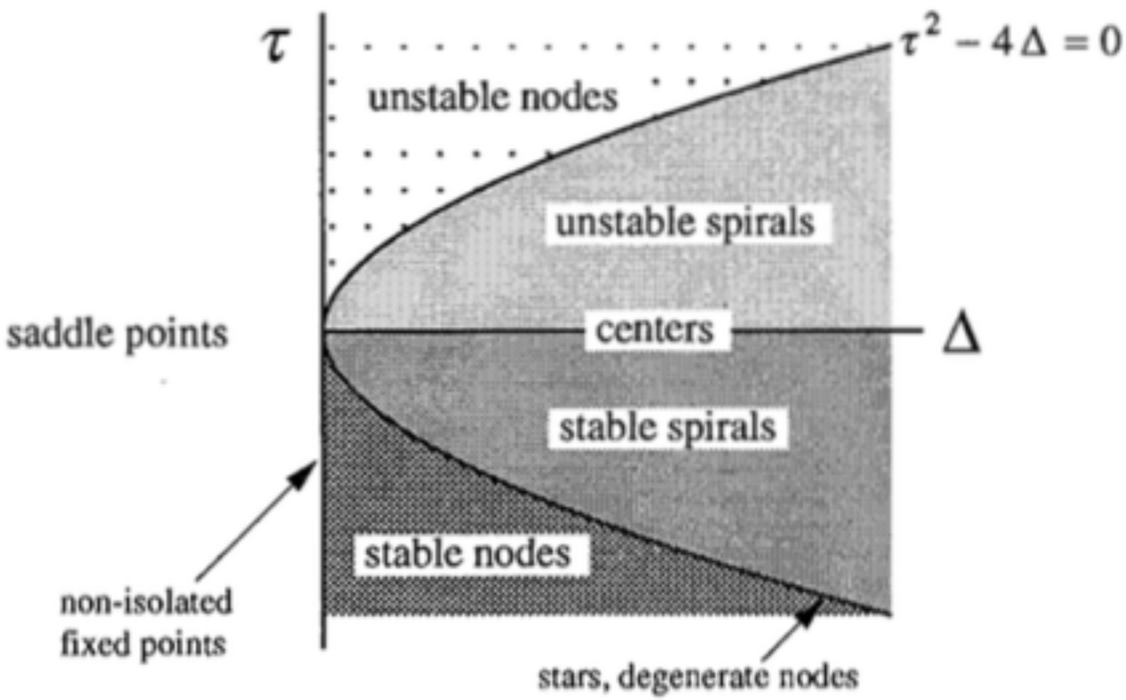
$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

…the **eigenvalues** of **A** are completely determined by the **trace** and **determinant**…

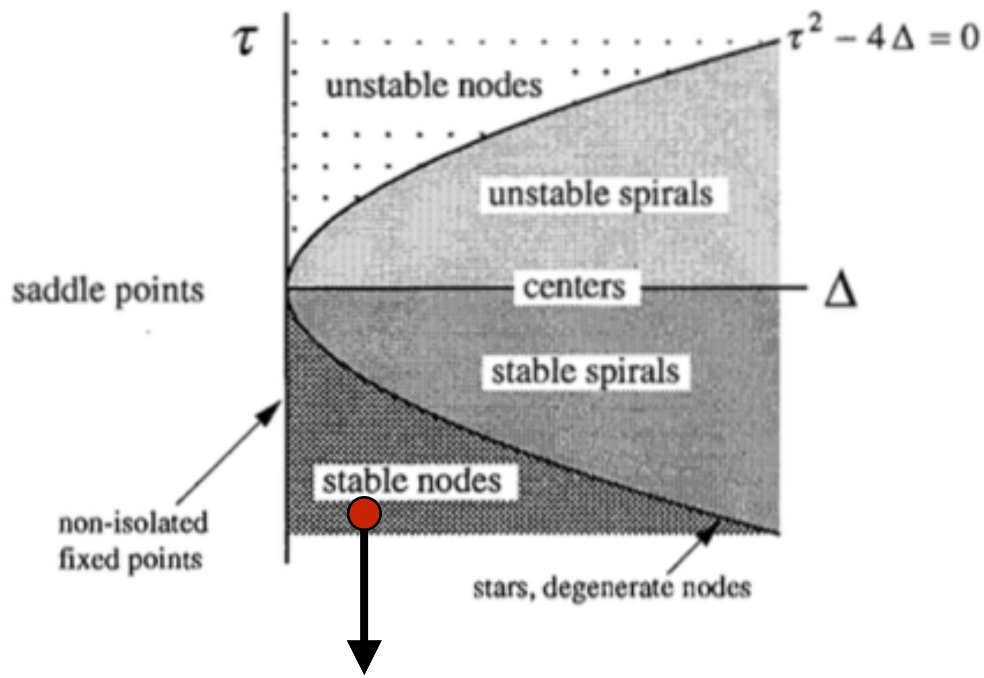
System behaviors: the second order linear case

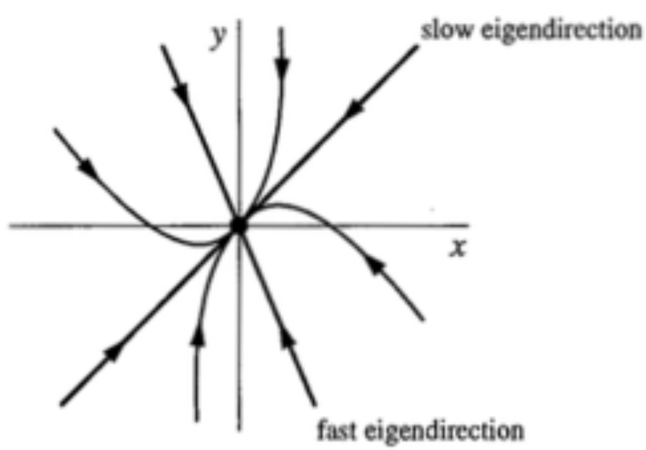
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$



…the **zoo of all possible behaviors** for a linear, second-order system

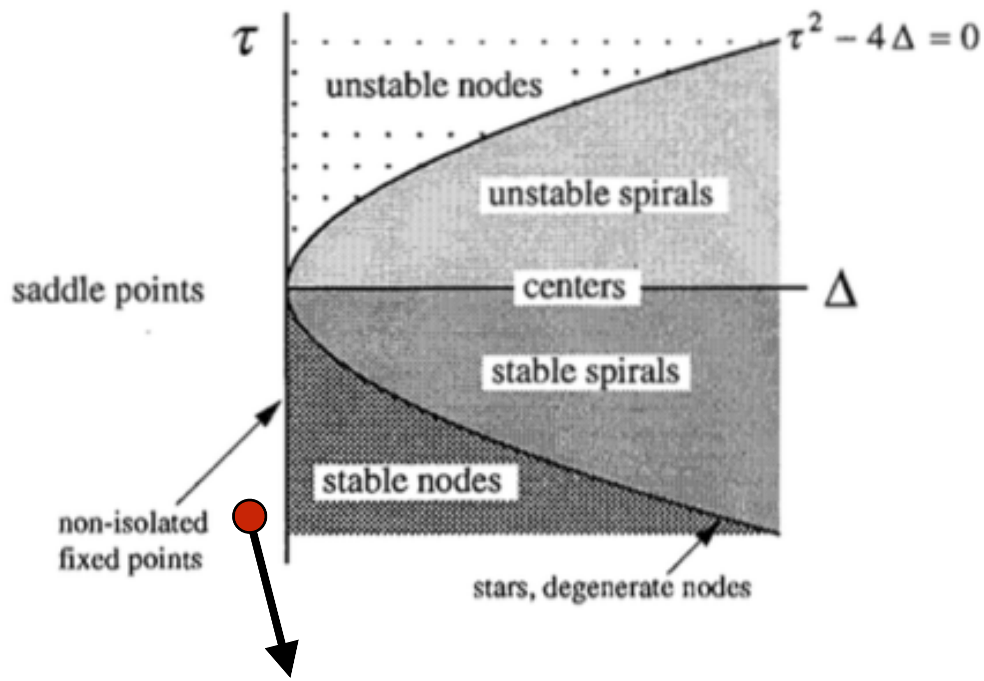
System behaviors: the second order linear case $\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$

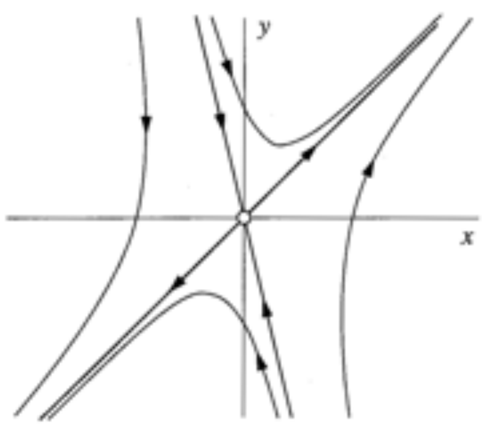




…stable nodes

System behaviors: the second order linear case $\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$

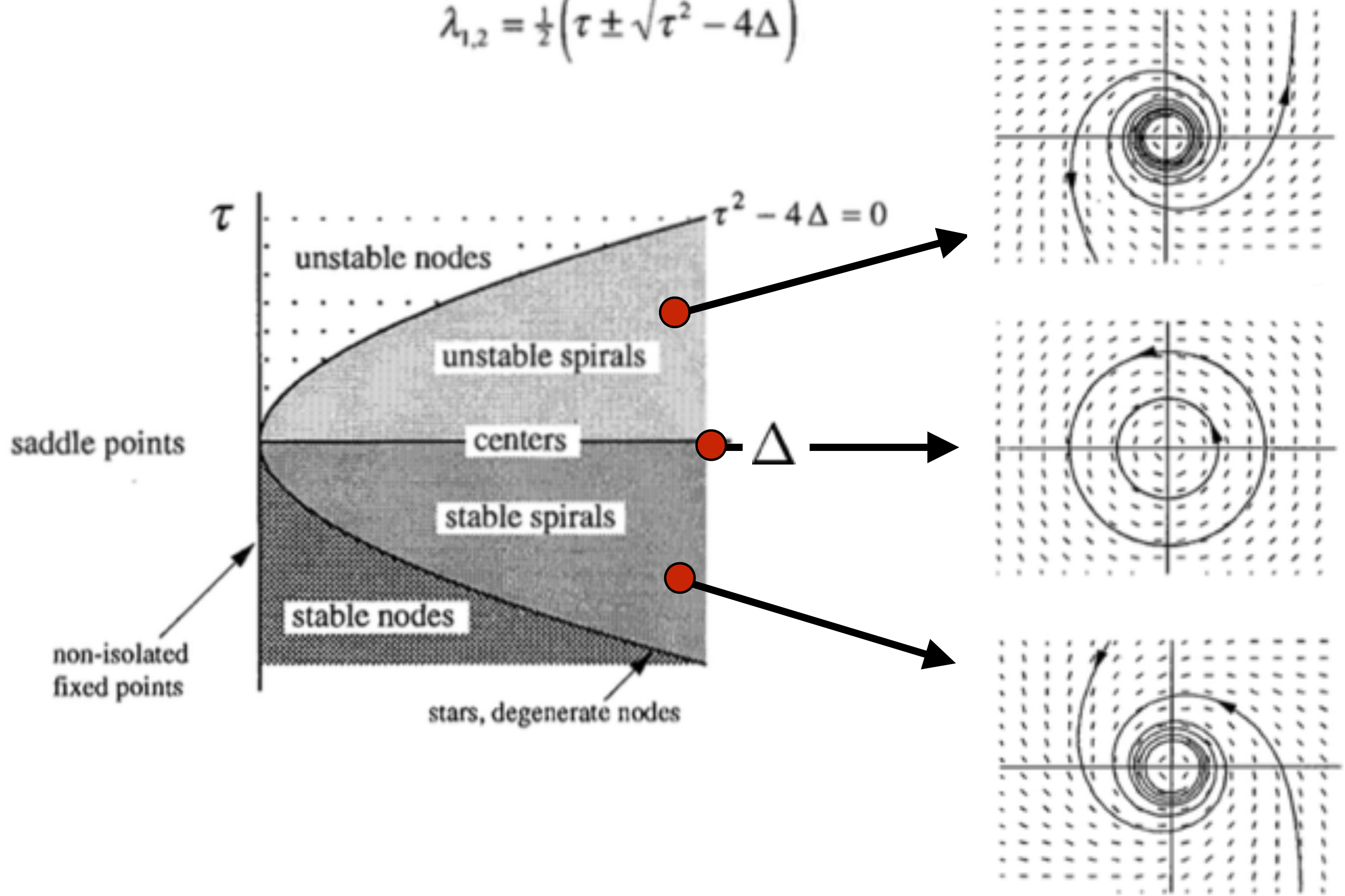




…**saddle** nodes

System behaviors: the second order linear case

$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$



…and **spiral** nodes when eigenvalues are complex numbers

System behaviors: the second order linear case

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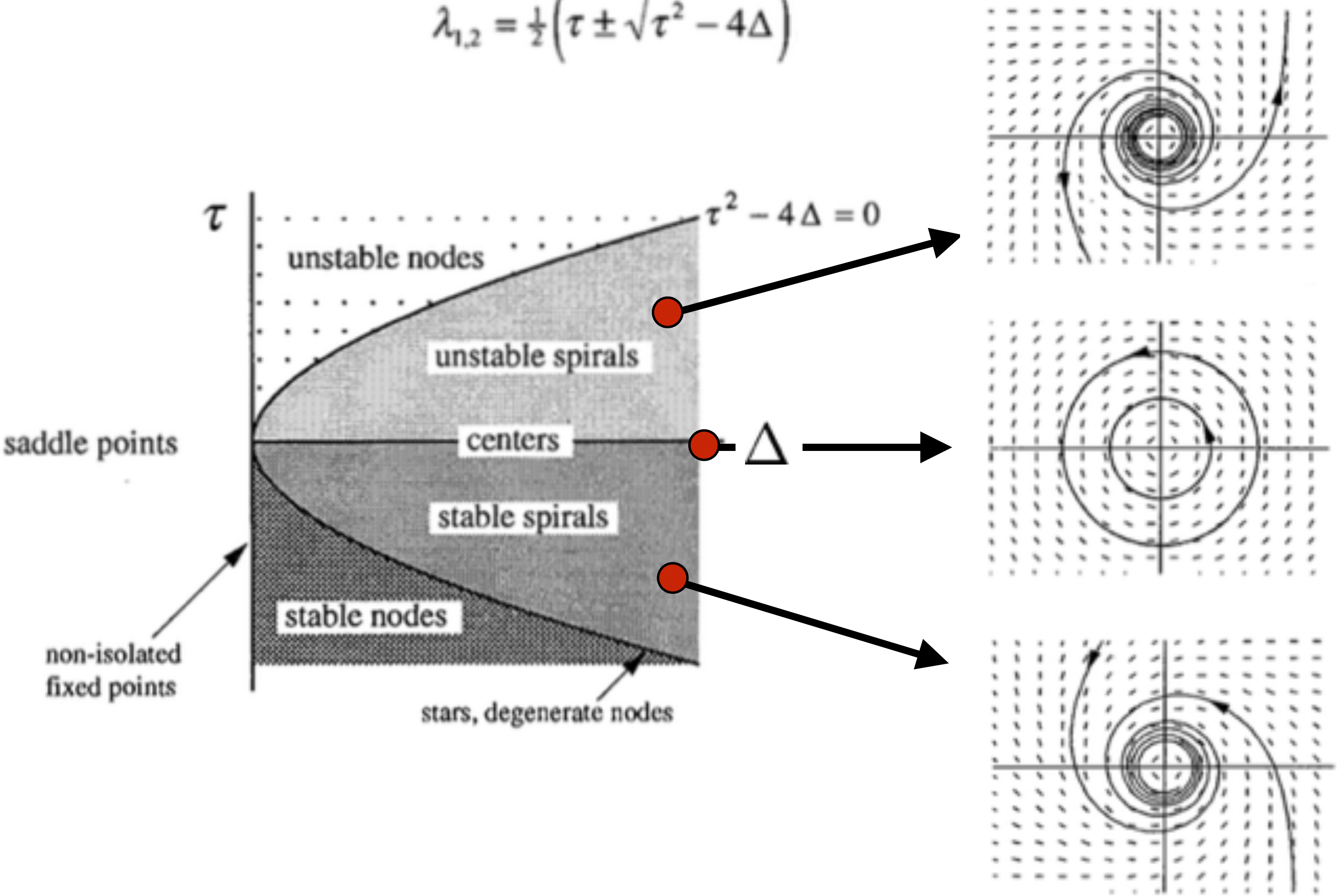
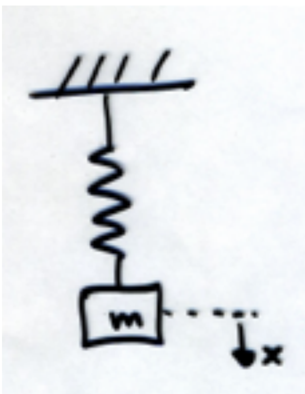


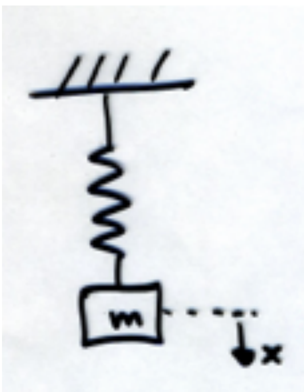
$$e^{x+iy} = e^x[\cos y + i\sin y]$$ ← …both **spiral** nodes when eigenvalues are complex numbers

Seeing behaviors: the linear harmonic oscillator



Often, it is hard to get analytic solutions. We need a way of **"seeing" system behavior**….

Seeing behaviors: the linear harmonic oscillator



Now, the equation of motion is:

$F = ma$, or...

$-kx = m\ddot{x}$

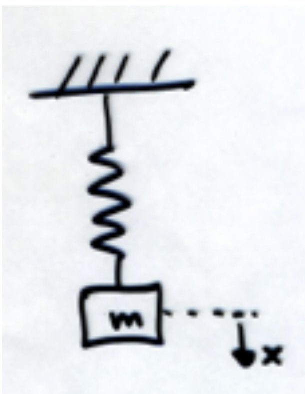
Remember that for a Hooke spring,

$F = -kx$, and

$\dot{x} \equiv \frac{dx}{dt} = v$

$\ddot{x} \equiv \frac{d^2x}{dt^2} = a$

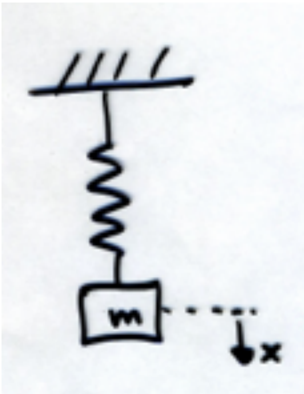
Seeing behaviors: the linear harmonic oscillator



So… $\boxed{m\ddot{x} + kx = 0}$

We can **re-write** this equation….

Seeing behaviors: the linear harmonic oscillator

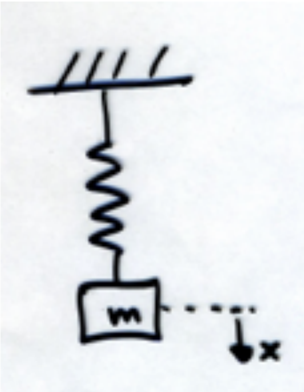


So... $\boxed{m\ddot{x} + kx = 0}$

$$\dot{x} = v$$
$$\dot{v} = -\frac{k}{m}x$$

The first equation is just the definition of velocity. The second equation is $m\ddot{x} + kx = 0$ re-written in terms of v.

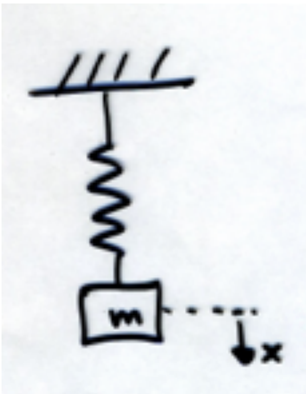
Seeing behaviors: the linear harmonic oscillator



To simplify, we define $\omega^2 = \frac{k}{m}$. So..

$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2 x \end{aligned}}$$

Seeing behaviors: the linear harmonic oscillator

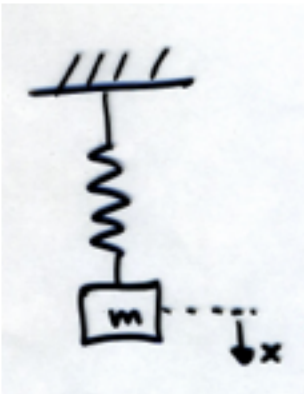


$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2 x \end{aligned}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

We know how to **analyze the behavior**, right?

Seeing behaviors: the linear harmonic oscillator



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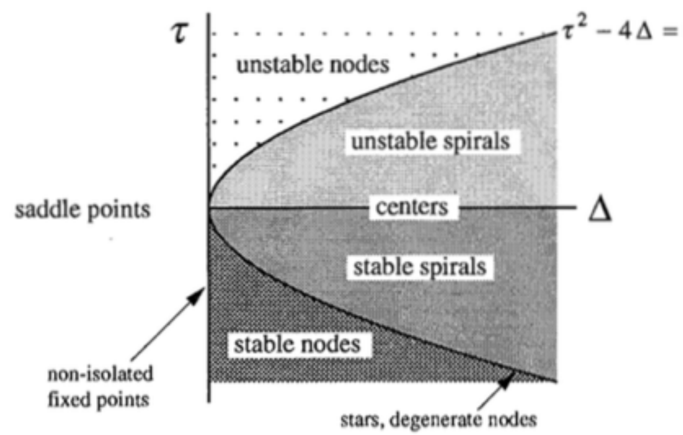
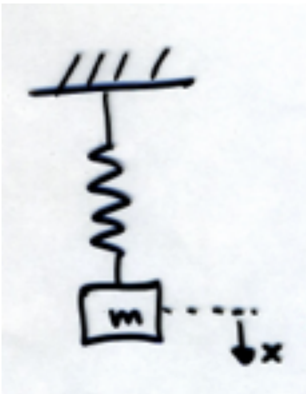
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

$$\downarrow$$



Seeing behaviors: the linear harmonic oscillator



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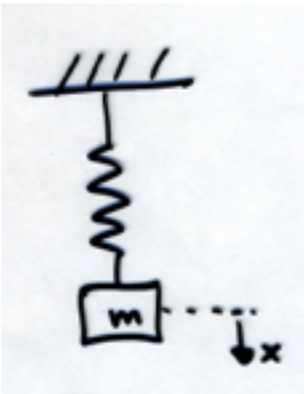
Trace = 0

Det = ω^2

So... $\lambda_{1,2} = \frac{1}{2}\left[0 \pm \sqrt{0 - 4\omega^2}\right]$

$= \pm i\omega$

Seeing behaviors: the linear harmonic oscillator



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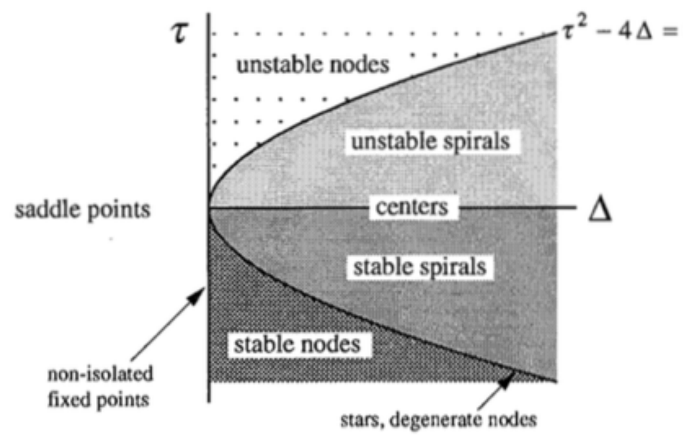

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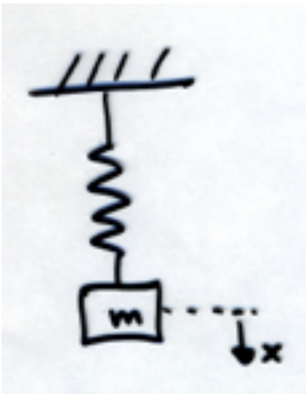
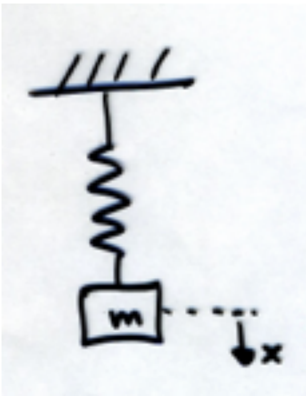


$$\boxed{\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\omega^2 x\end{aligned}}$$

Thus, for every (x,v), this system of equations assigns a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$. This makes a vector field ...

....for this system, (x,v) represents a 2D **"phase space"** in which we can see the behavior of the system intuitively.

Seeing behaviors: the linear harmonic oscillator



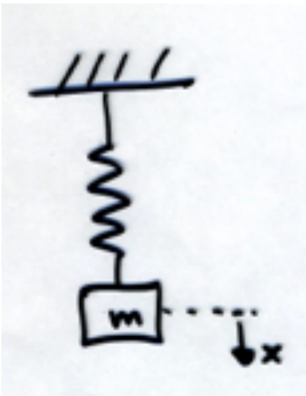
$$\boxed{\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\omega^2 x\end{aligned}}$$

plot the system **nullclines**….

$$(\dot{x},\dot{v}) = (0,-\omega^2 x)$$

$$(\dot{x},\dot{v}) = (v,0)$$

Seeing behaviors: the linear harmonic oscillator



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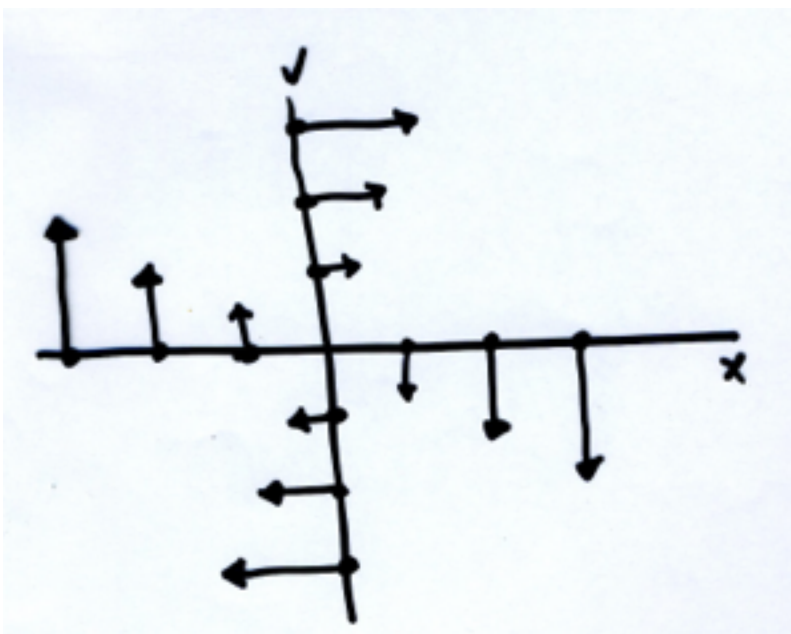

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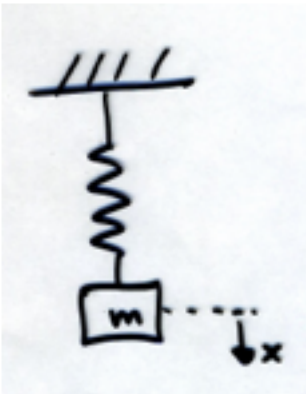
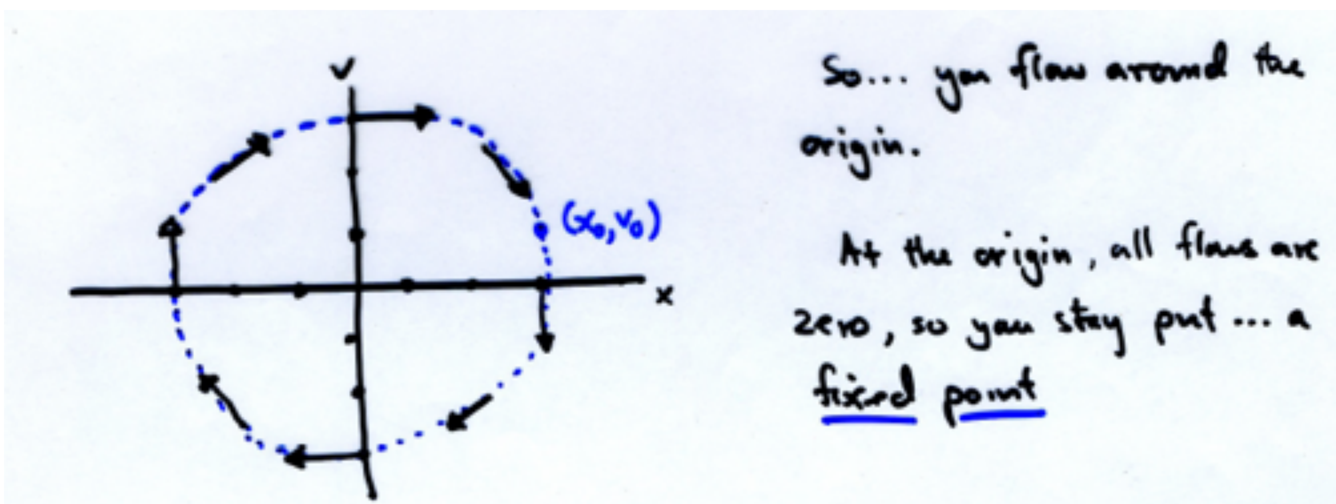
Seeing behaviors: the linear harmonic oscillator





This is called a “**phase portrait**”…a way of seeing system dynamics.

A summary….

Linear systems are:

(1) **decomposable**, such that high-order systems are combinations of first-order systems. This is the concept that the behavior of the whole is predictable from knowledge of the behavior of the underlying parts.

(2) **understandable**; their behavior can be mapped through a study of their so-called eigenfunctions. This is the concept that one can “understand” the properties of linear systems by sketching the behavior of these eigenfunctions.

(3) **simple**; these systems show single fixed points…whether stable or unstable

A **non-linear oscillator...**

$$\boxed{\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0}$$ (the van der Pol oscillator)

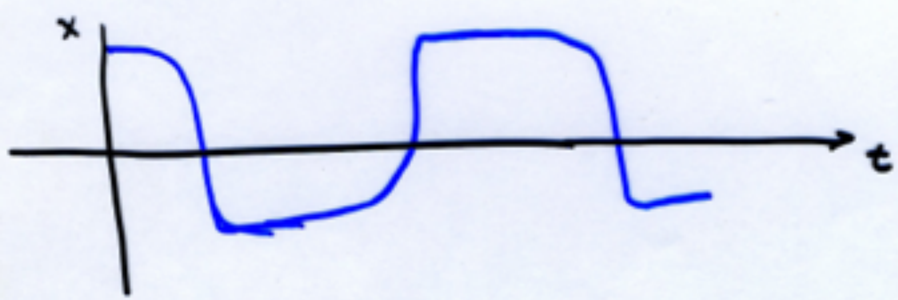
Here is the **non-linearity**….with mu controlling the degree of non-linearity.

A **non-linear oscillator...**

$$\boxed{\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0}$$ (the van der Pol oscillator)

For $\mu >> 1$, this is the strongly non-linear limit. There is a position-dependent damping term $\mu(x^2-1)\dot{x}$. This acts like positive damping for $|x| > 1$ to cause oscillations to decay, but acts like negative damping for $|x| < 1$ to build oscillations up.

It causes so-called "<u>relaxation oscillations</u>" ...



A **non-linear oscillator...**

$$\boxed{\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0}$$

Lets write the equation in the usual $\dot{x} =$, $\dot{y} =$ way to make a phase plane portrait...

A little re-definition of variables... note that

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0 \qquad (1)$$

$$\ddot{x} + \mu(x^2-1)\dot{x} = \frac{d}{dt}\left(\dot{x} + \mu\left[\tfrac{1}{3}x^3 - x\right]\right) \qquad (2)$$

$$F(x) = \tfrac{1}{3}x^3 - x \qquad (3)$$

$$w = \dot{x} + \mu F(x) \qquad (4)$$

So...

$$\dot{w} = \ddot{x} + \mu(x^2-1)\dot{x} = -x \qquad \text{using (4), (2), (1)}$$

Re-writing the equations in a **more intuitive** way....

A **non-linear oscillator...**

$$\boxed{\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0}$$

we can now rewrite the van der Pol eqn (1) as..

$$\boxed{\begin{aligned} \dot{x} &= \omega - \mu F(x) \\ \dot{\omega} &= -x \end{aligned}}$$

A **non-linear oscillator...**

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One other convenience... set $y = \frac{\omega}{\mu}$. Then...

$$\boxed{\begin{aligned}\dot{x} &= \mu[y - F(x)]\\ \dot{y} &= -\frac{1}{\mu}x\end{aligned}}$$

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Now, to see behavior of our system, we sketch the so-called "nullclines" of the system... the equations corresponding to $\dot{x}=0$ and $\dot{y}=0$.

The van der Pol oscillator....

$$\boxed{\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0}$$

$$\dot{x} = \mu[y - F(x)]$$
$$\dot{y} = -\frac{1}{\mu}x$$

Nullclines are . . .

$x = 0$, and . . .

$$y = F(x)$$
$$= \frac{1}{3}x^3 - x \quad \text{[a cubic function]}$$

The van der Pol oscillator....

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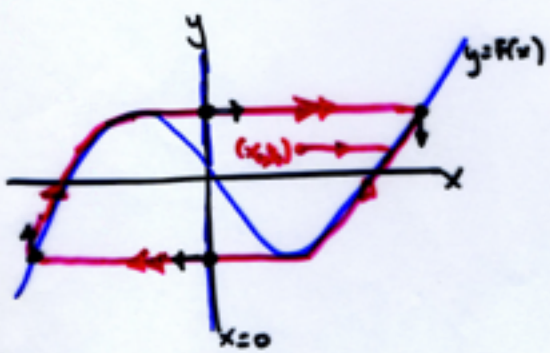
Plots...



The van der Pol oscillator....

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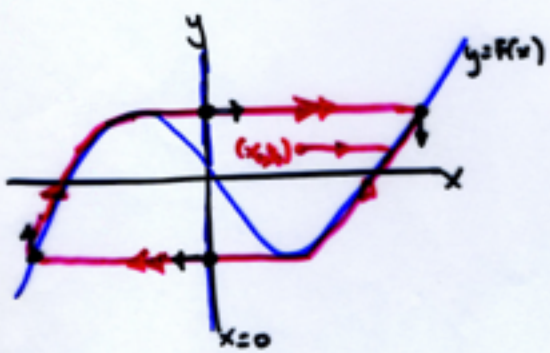

① Take (x_0, y_0) far from nullclines. ... say $y - F(x) \sim O(1)$

If so... $|\dot{x}| \sim O(\mu) \gg 1$ (because $\mu \gg 1$).

$|\dot{y}| \sim O(\mu^{-1}) \ll 1$

So ... the flow is all horizontal and <u>fast</u>!

② Now if $y - F(x) \sim O(\mu^{-2})$ ... that is, close to cubic nullcline,

$|\dot{x}| \sim O(\mu^{-1})$

$|\dot{y}| \sim O(\mu^{-1})$

... flows equal in both directions ... and <u>slow</u>!

The van der Pol oscillator....

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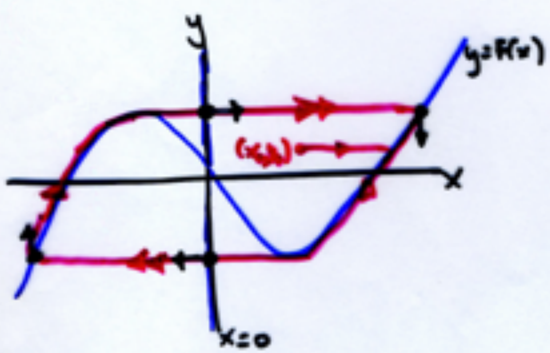

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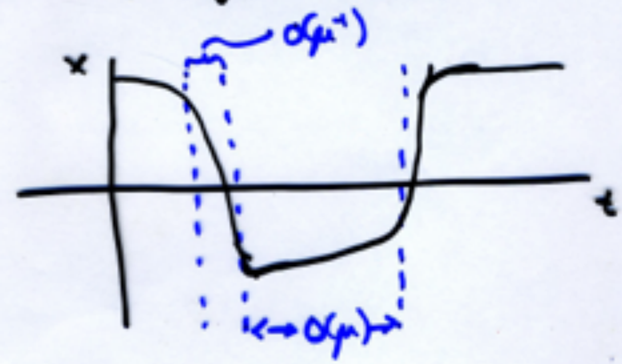
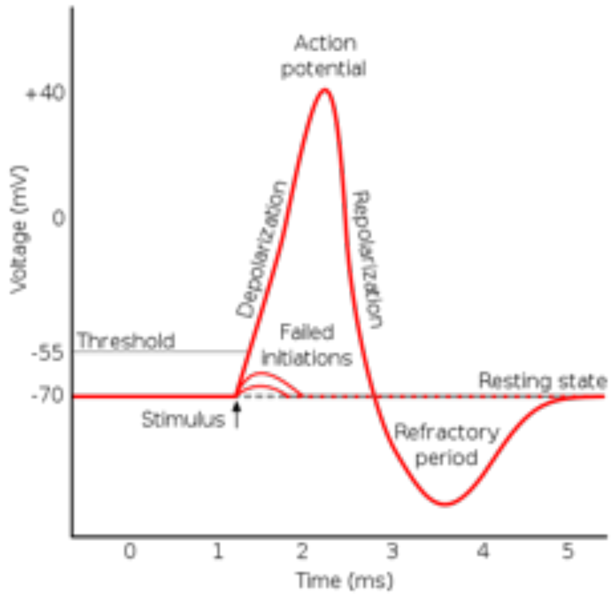
Thus plotting $x(t)$:



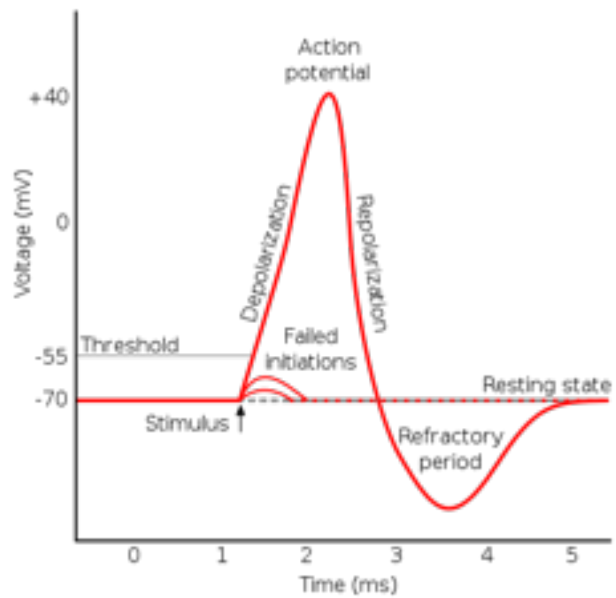
A **non-linear oscillator...**

We will study the behavior of these systems in more detail next time, but as a preview, the basic model for the **neuronal action potential i**s only a slight variation on the van der Pol oscillator…



A **non-linear oscillator...**

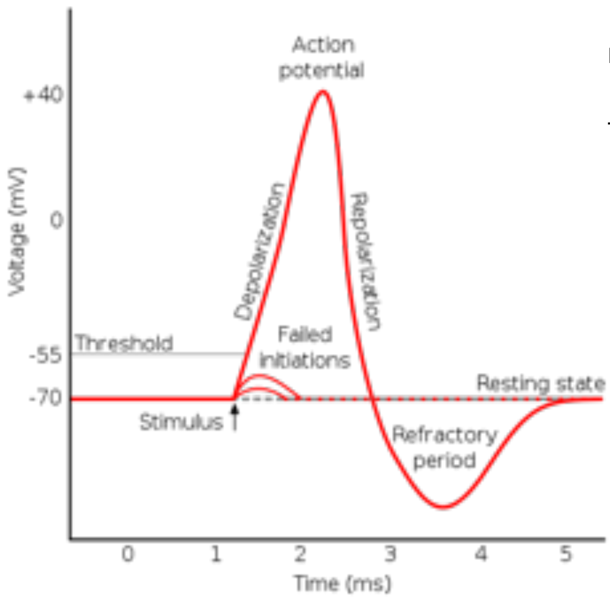
We will study the behavior of these systems in more detail next time, but as a preview, the basic model for the **neuronal action potential i**s only a slight variation on the van der Pol oscillator…



a regenerative, **non-linear activation** with a sharp **threshold**, an all-or-nothing character, and a **refractory period** afterwards…

A **non-linear oscillator...**

Hodgkin-Huxley (1952)



membrane pot
$$\frac{dV}{dt} = C_{\mathrm{Na}} m^3 h (E_{\mathrm{Na}} - V) + C_{\mathrm{K}} n^4 (E_{\mathrm{K}} - V) + C_{\mathrm{leak}} (V_{\mathrm{rest}} - V) + I_{\mathrm{inj}}(t)$$

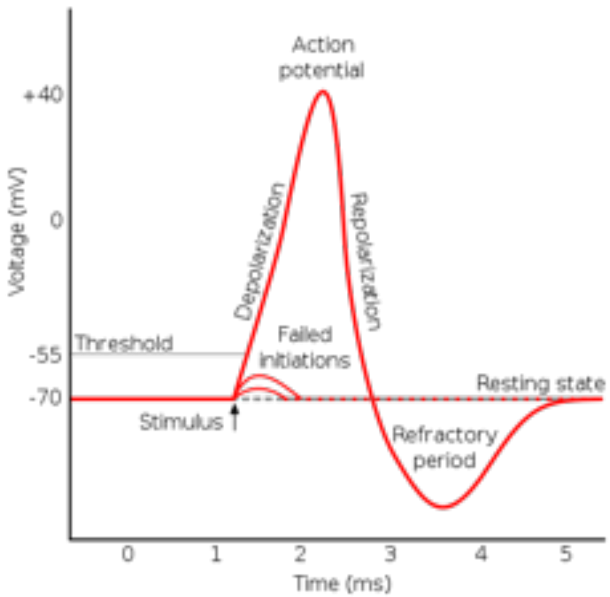
fast Na+ flux
$$\frac{dm}{dt} = \alpha_m(V)(1-m) - \beta_m(V)\, m$$

slow Na+ flux
$$\frac{dh}{dt} = \alpha_h(V)(1-h) - \beta_h(V)\, h$$

slow K+ flux
$$\frac{dn}{dt} = \alpha_n(V)(1-n) - \beta_n(V)\, n$$

but, **Fitzhugh and Nagumo** simplified this 4D set of equations….

A **non-linear oscillator...**



Fitzhugh-Nagumo (1962)

membrane pot $$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I$$

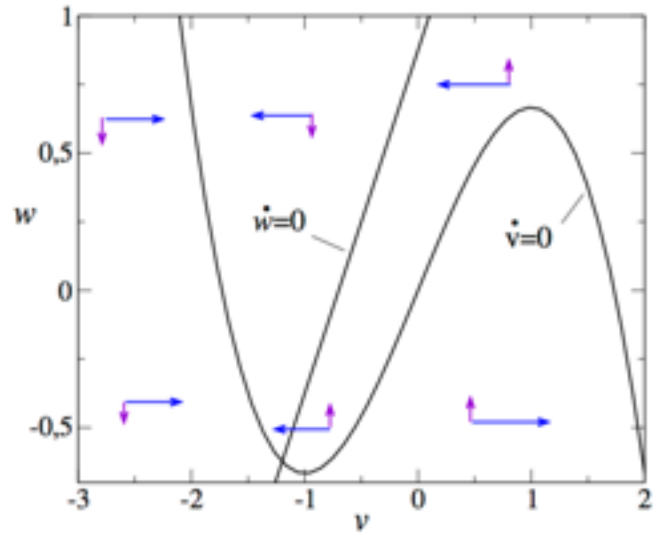
slow K+ flux $$\frac{dw}{dt} = \frac{1}{\tau}(v + a - bw)$$

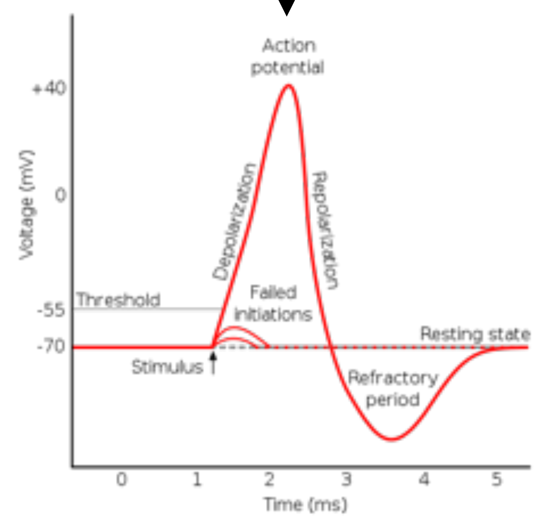
this is essentially the van der Pol oscillator, with **one difference**….

A **non-linear oscillator...**

membrane pot
$$\frac{\mathrm{d}v}{\mathrm{d}t} = v - \frac{v^3}{3} - w + I$$

slow K+ flux
$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{1}{\tau}(v + a - bw)$$





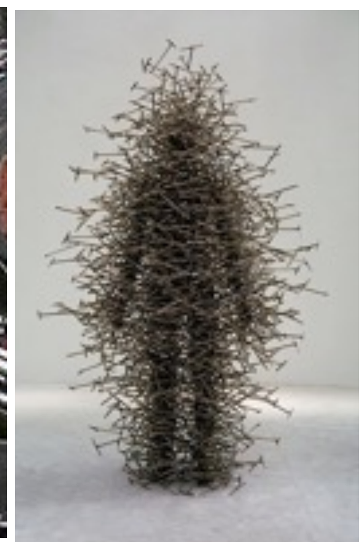
the linear term to the w nullcline provides for **thresholded oscillation**….you will see next time

A 1D **discrete-time non-linear system**

$$\boxed{f(s) = g\, s\, (1-s)}$$

A seemingly innocuous thing….the so-called
logistic equation

But led to principles have **broad application** in both basic and applied science….and art, social science, and the popular media.



A 1D **discrete-time non-linear system**

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A seemingly innocuous thing….the so-called
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$$\boxed{f(s) = g\, s\,(1-s)}$$

Lets see.... using a graphical method called an iterative map:

This plots the value of a function against its previous value....

$$f(s) = F\{f(s-1)\}$$

An **iterative map** gives the current value of a system as a function of its previous value…

$$f(s) = F\{f(s-1)\}$$

<u>Example</u>:

$$f(s+1) = f(s) + c \quad ; \quad \text{say } f(0) = S_0$$

$$f(0) = S_0$$

$$f(1) = S_0 + c$$

$$f(2) = S_0 + c + c = S_0 + 2c$$

$\vdots$

$$f(s) = F\{ f(s-1) \}$$

Example:

$f(s+1) = f(s) + c$; Say $f(0) = S_0$

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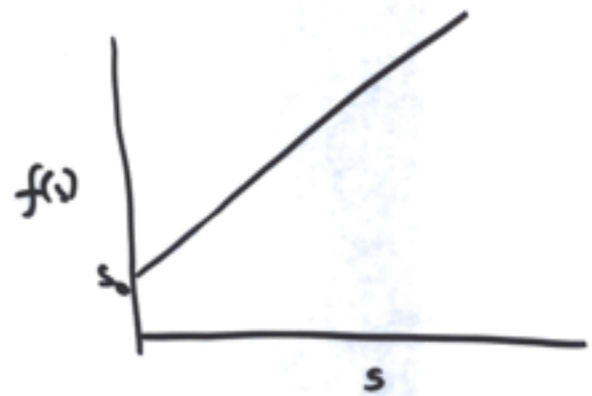
$\vdots$



The equation for constant velocity motion….

Note that these maps are discrete time mappings! Also, we are now discussing just 1-D maps. Why 1D? only one variable we are following:

$$f(s) = F\{f(s-1)\}.$$

Fixed points….a more formal treatment

Fixed points:

Say a certain value of x satisfies the rule that

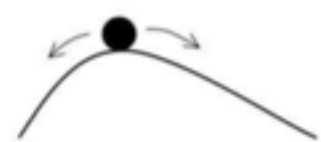
$$f(x^*) = x^*$$

Then this value of x is called a fixed point, because the orbit stops at x^* for all future values of x.

Stability of the fixed point:

To determine stability, the idea is to cause a small perturbation and ask whether the orbit is attracted back to x^* or is repelled.

stable steady state



unstable steady state

Fixed points….a more formal treatment

So consider…

$$x_n = x^* + \underbrace{\eta_n}_{\text{perturbation}}$$

$$x_{n+1} = x^* + \eta_{n+1} = f(x^* + \eta_n)$$

$$= f(x^*) + f'(x^*)\eta_n + \underbrace{O(\eta_n^2)}_{\text{What is this?}}$$

Fixed points….a more formal treatment

So consider…

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$$= f(x^*) + f'(x^*)\eta_n + O(\eta_n^2)$$

What is this? Well, the **higher order stuff,** which we will conveniently ignore….

Fixed points….a more formal treatment

So consider…

$$x_n = x^* + \underbrace{\eta_n}_{\text{perturbation}}$$

$$x_{n+1} = x^* + \eta_{n+1} = f(x^* + \eta_n)$$

$$= f(x^*) + f'(x^*)\eta_n + \underline{O(\eta_n^2)}$$

But $f(x^*) = x^*$, so…

$$\eta_{n+1} = f'(x^*)\eta_n$$

Call the "multiplier" of the perturbation $\lambda = f'(x^*)$. Then

$$\eta_1 = \lambda\eta_0$$
$$\eta_2 = \lambda^2\eta_0$$
$$\vdots$$
$$\eta_n = \lambda^n\eta_0$$

Fixed points….a more formal treatment

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$$\vdots$$
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So… if $|\lambda| < 1$, then $\eta_n \to 0$ as $n \to \infty$ [stable fixed pt]

if $|\lambda| > 1$, then $\eta_n \to \infty$ as $n \to \infty$ [unstable]

Fixed points….a more formal treatment

Example: $x_{n+1} = x_n^2$

Fixed points?

Fixed points….a more formal treatment

Example: $x_{n+1} = x_n^2$

Fixed points? 0 or 1.

Stability?

Fixed points….a more formal treatment

Example: $x_{n+1} = x_n^2$

Fixed points? 0 or 1.

Stability?

$\lambda = f'(x^*) = 2x^*$

Fixed points….a more formal treatment

Example: $x_{n+1} = x_n^2$

Fixed points? 0 or 1.

Stability?

$\lambda = f'(x^*) = 2x^*$

So…the fixed point at zero is **stable**, and the one at 1 is **not**.

We will do a similar analysis for the logistic equation soon.

Iterative Maps

1. $x_{n+1} = x_n + c$, where c is a constant. Say $x_0 = 3.5$ ...
$c = 10$

Iterative Maps

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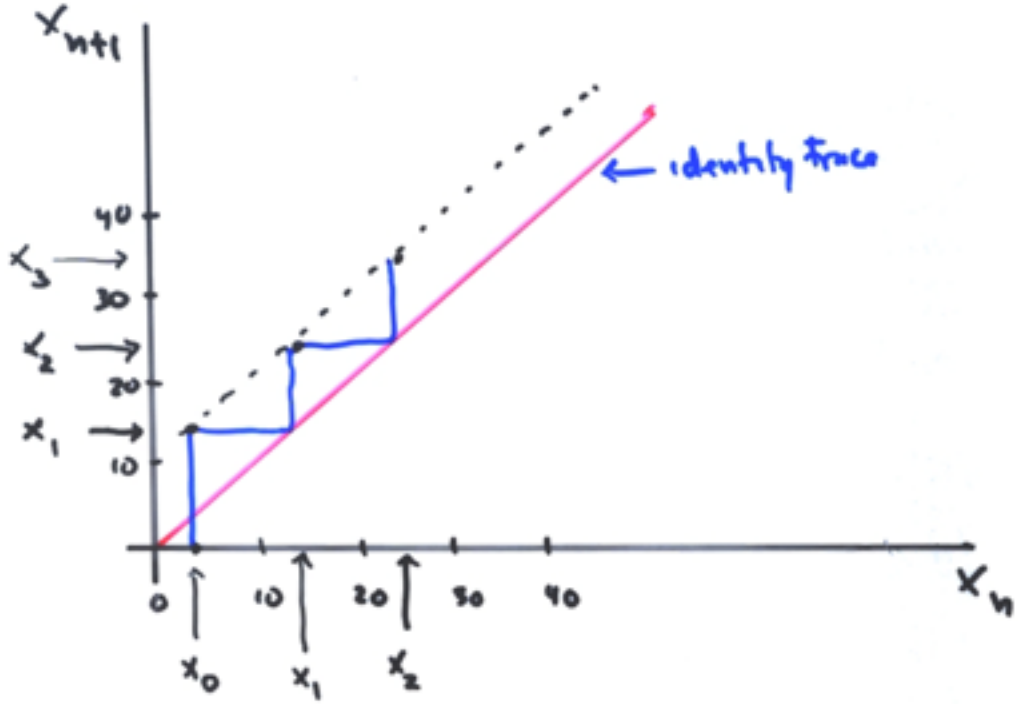
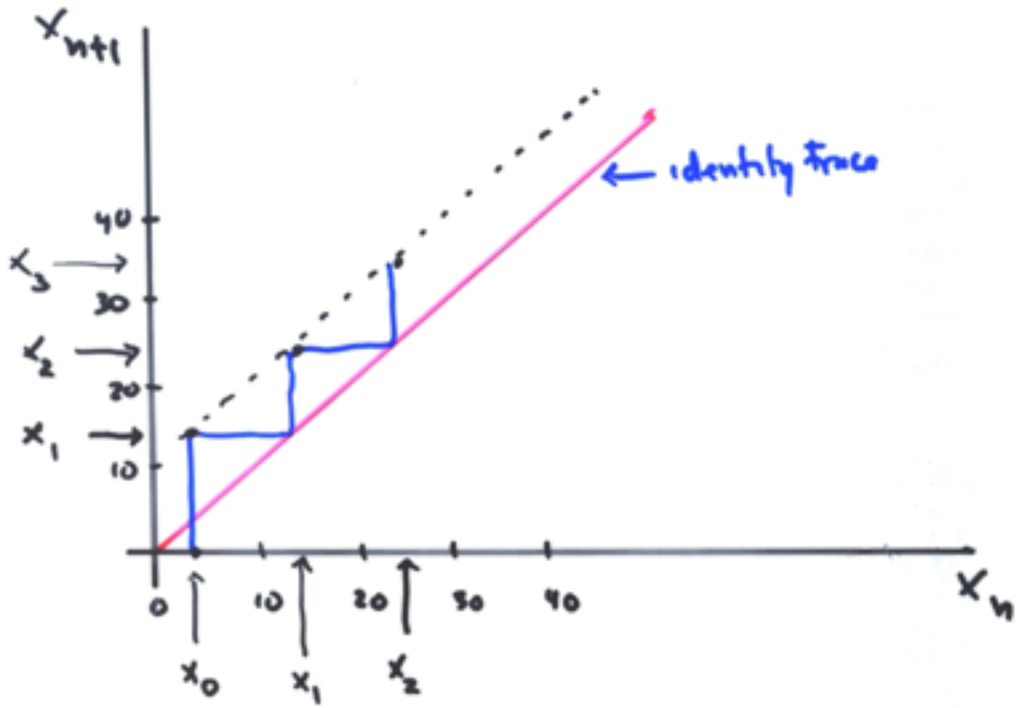


Just a way of plotting the "**orbit**"…or the behavior of the equation.

Iterative Maps

1 $x_{n+1} = x_n + c$, where c is a constant. Say $x_0 = 3.5$ ...

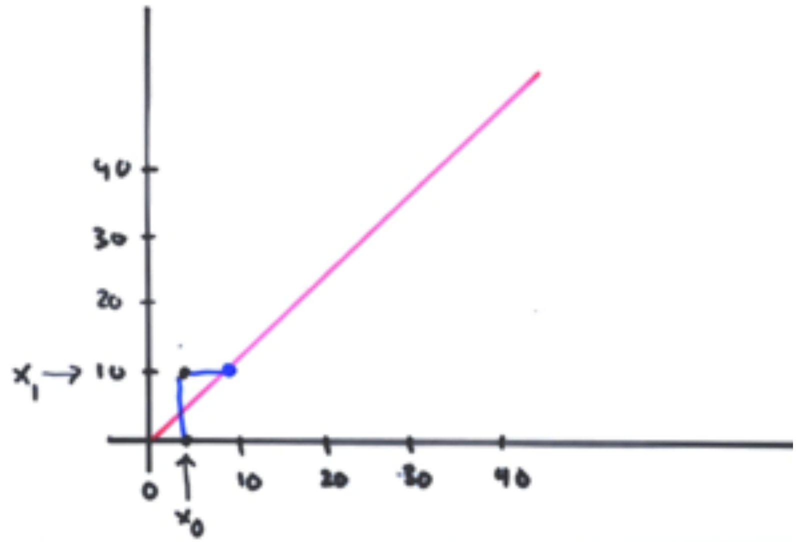
$c = 10$



In this plot, what would a **fixed point** be? Are there any?

Iterative Maps

(2) $x_{n+1} = c$, say $x_0 = 3.5$, $c = 10$



This is a **really boring** equation….

Iterative Maps

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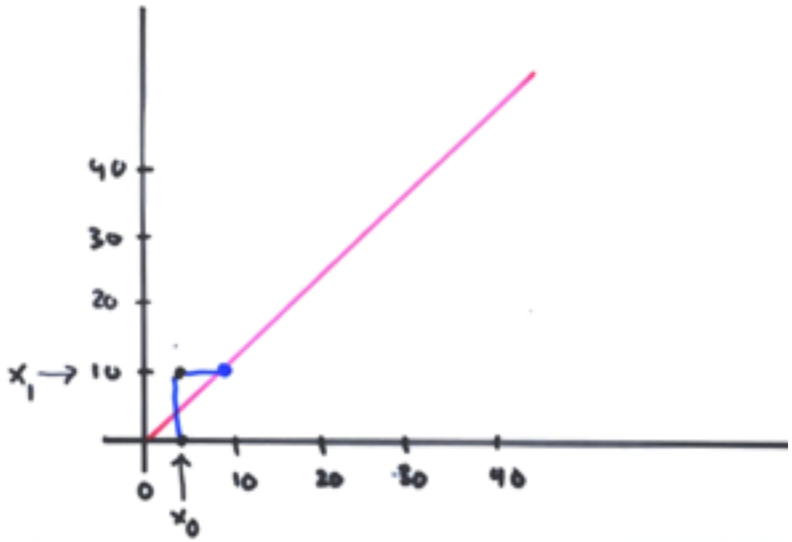
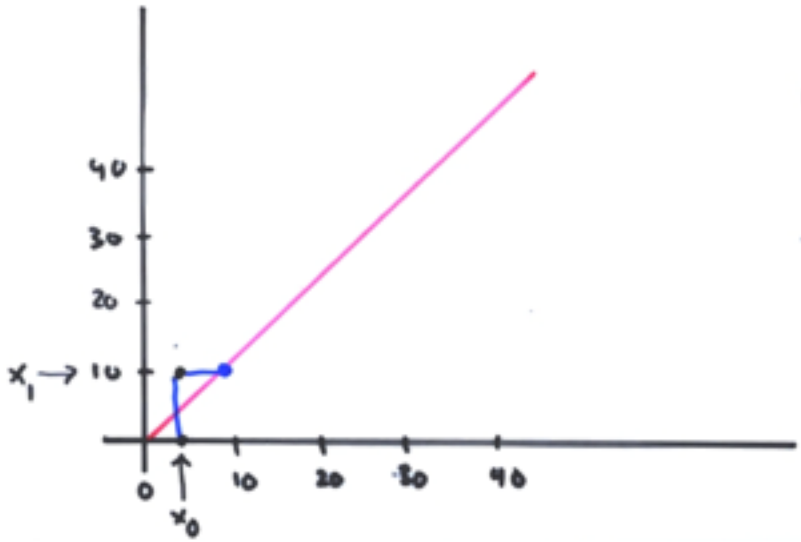


This is a **really boring** equation….but it does have a fixed point! What about stability of the fixed point?

Iterative Maps

(2) $x_{n+1} = c$, say $x_0 = 3.5$, $c = 10$



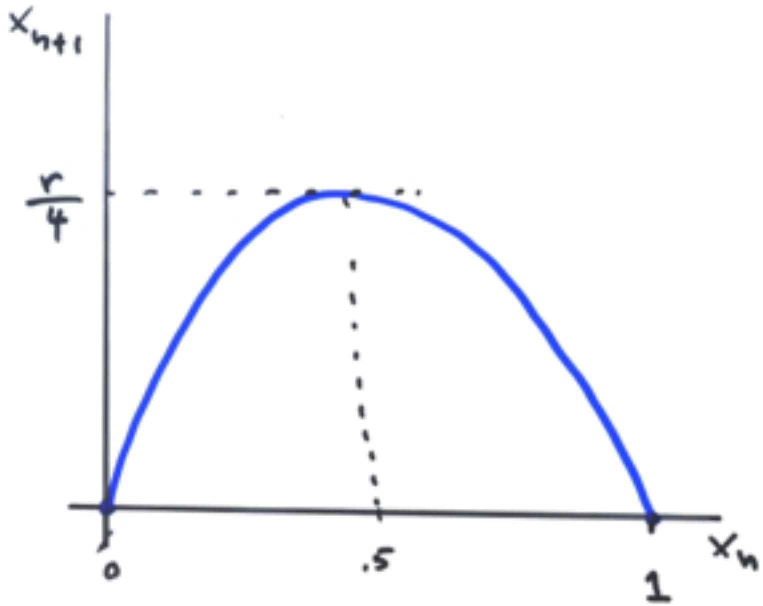
well, $x = 10$ is a fixed point.
stable ?

The places where a function crosses the identity trace are the fixed points.

Iterative Maps

3 So … now for the so-called logistic equation

$$x_{n+1} = r\,x_n(1-x_n)$$



Ok…..but where is the **identity trace** relative to the curve?

Iterative Maps

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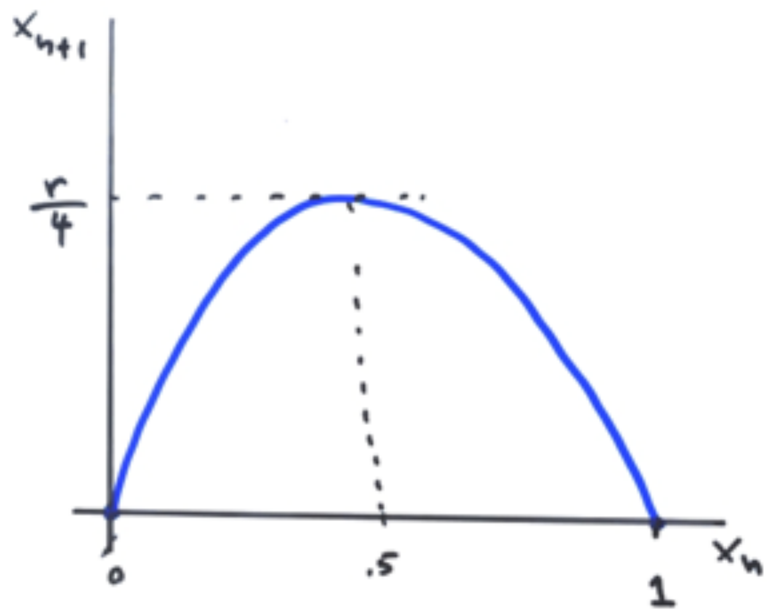
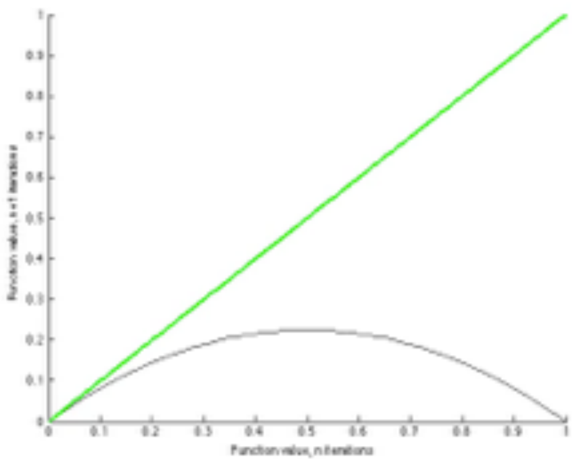


Ok…..but where is the **identity trace** relative to the curve? Well….it depends on **r**…

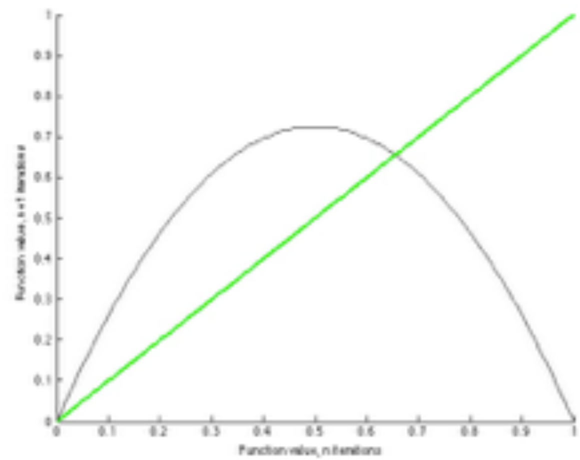
Iterative Maps

$$y(n+1) = ry(n)(1-y(n))$$

r = 0.9



r = 2.9



Ok…..now, where are the **fixed points** and what about **stability**?

Analysis of the logistic map.

$$x_{n+1} = r\,x_n(1-x_n)$$

$0 \le x_n \le 1$

$0 \le r \le 4$ ... the interesting range.

Analysis of the logistic map.

$$x_{n+1} = r\,x_n(1-x_n) \qquad 0 \le x_n \le 1$$

$$0 \le r \le 4$$

① Find fixed points.

$$x^* = f(x^*) = r\,x^*(1-x^*)$$. where is this true?

Analysis of the logistic map.

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$$0 \le r \le 4$$

① Find fixed points.

$$x^* = f(x^*) = r x^* (1 - x^*) \quad .$$ where is this true?

$$x^* = 0$$

$$x^* = 1 - \frac{1}{r}$$

Analysis of the logistic map.

$$x_{n+1} = r\,x_n(1-x_n) \qquad 0 \le x_n \le 1$$

$$0 \le r \le 4$$

② Stability

well... $\lambda = f'(x^*) = r - 2rx^*$

For $x^* = 0$, the origin is stable for $r<1$ (since $f'(x^*) = r$)

unstable for $r>1$

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For $x^* = 1 - \frac{1}{r}$, $f'(x^*) = 2 - r$. So...

stable for $1 < r < 3$

unstable for $r > 3$

$$x_{n+1} = r x_n (1 - x_n) \qquad 0 \le x_n \le 1$$

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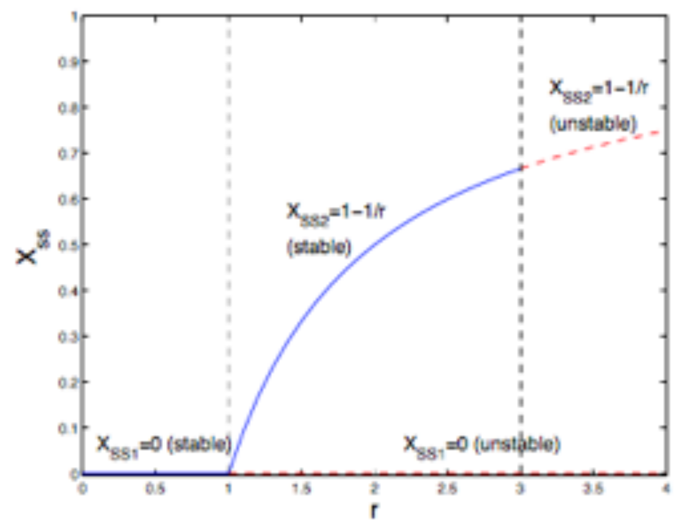

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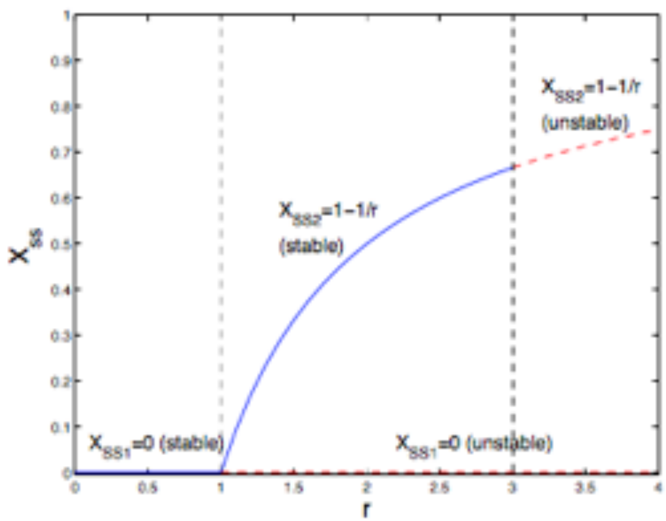
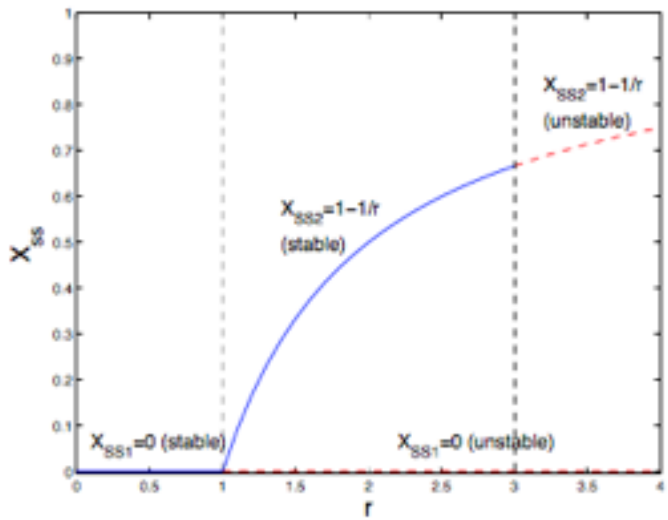
$$0 \le r \le 4$$



What happens for **r > 3**?

$$x_{n+1} = r\, x_n (1 - x_n) \qquad 0 \le x_n \le 1$$

$$0 \le r \le 4$$



What happens for **r > 3**?

Nature **261** 459–67 (1976)

Simple mathematical models with very complicated dynamics

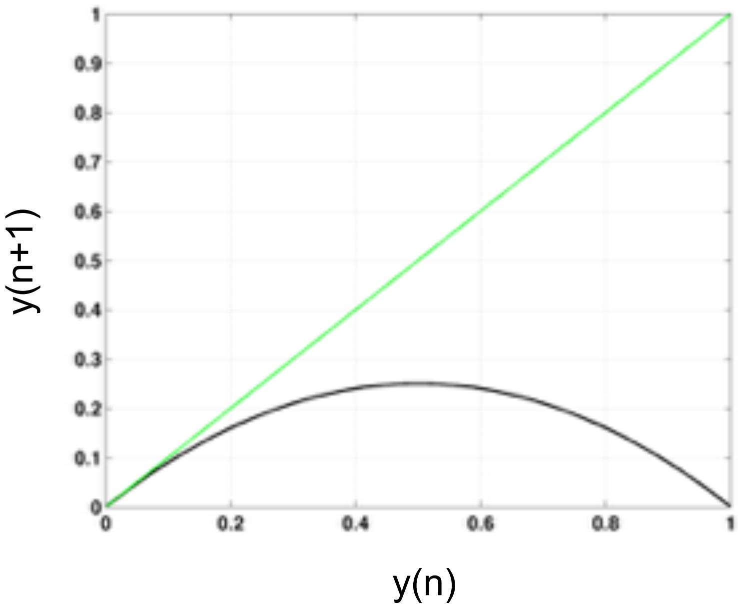
Robert M. May*

Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realised that simple nonlinear systems do not necessarily possess simple dynamical properties.

R. May., Nature (1976) 261, 459-67

The spectacular consequences of a small bit of non-linearity....

$$y(n+1) = ry(n)(1 - y(n))$$



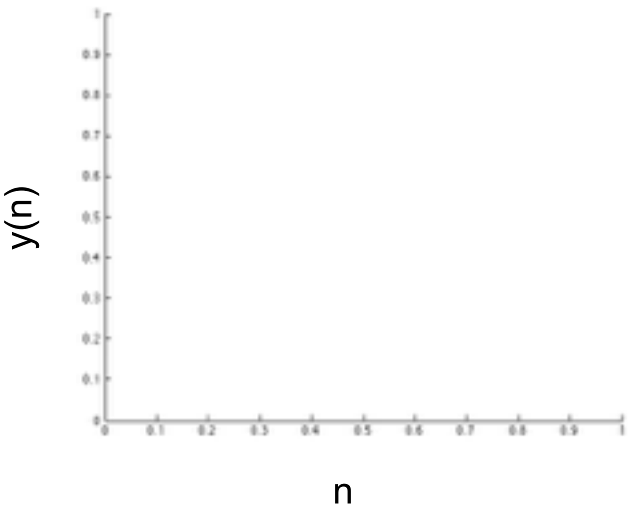
Let's look at the dynamics of this equation. We will start with y(0)=0.9, and consider 100 iterations at various values of r. Remember that r is basically the feedback strength in our small positive feedback reaction scheme....

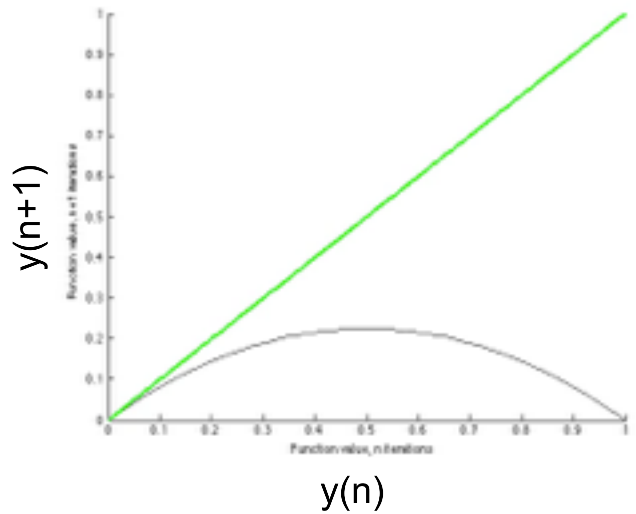
R. May., Nature (1976) 261, 459-67

The spectacular consequences of a small bit of non-linearity....

$$y(n+1) = ry(n)(1 - y(n))$$

y(0) = 0.9, r = 0.9





well... $\lambda = f'(x^*) = r - 2rx^*$

For $x^* = 0$, the origin is stable for $r < 1$ (since $f'(x^*) = r$)

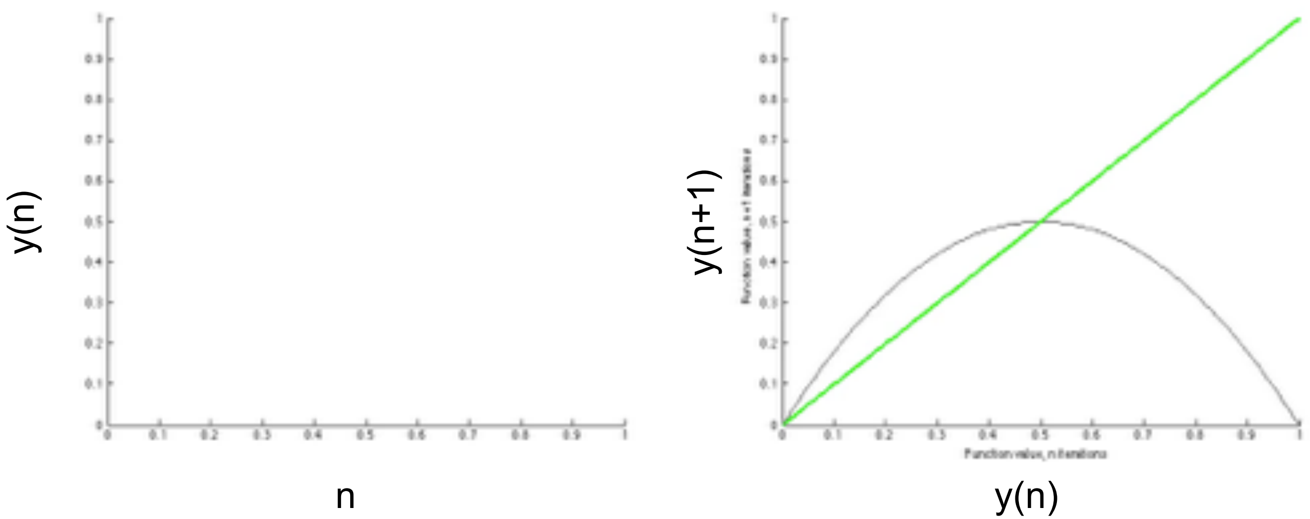
unstable for $r > 1$

R. May., Nature (1976) 261, 459-67

The spectacular consequences of a small bit of non-linearity....

$$y(n+1) = ry(n)(1 - y(n))$$

y(0) = 0.9, r = 2.0



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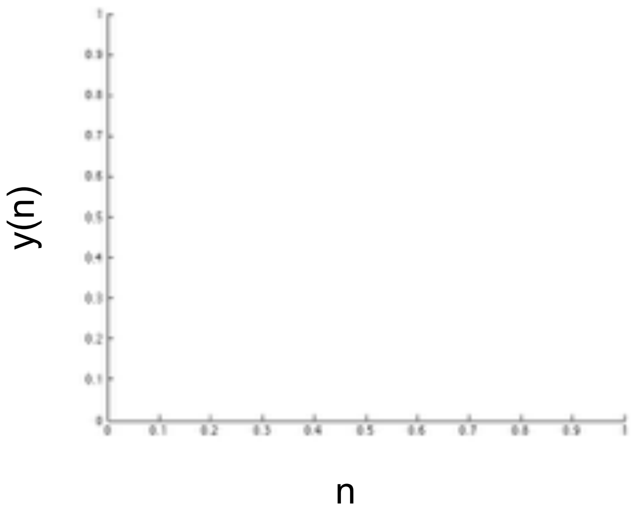
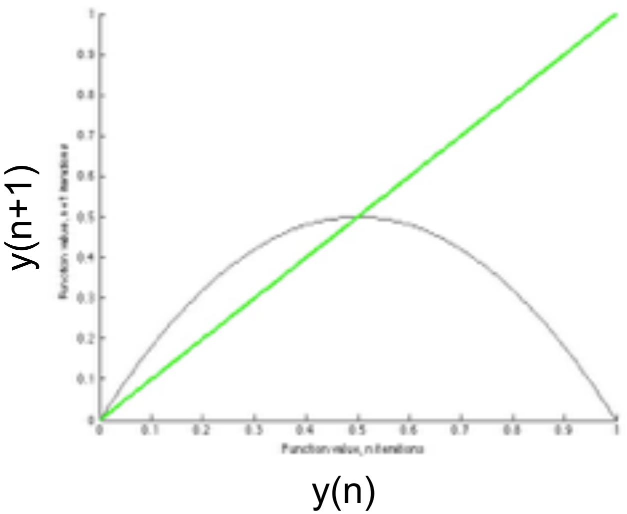




For $x^* = 1 - \frac{1}{r}$, $f'(x^*) = 2 - r$. So...

stable for $1 < r < 3$

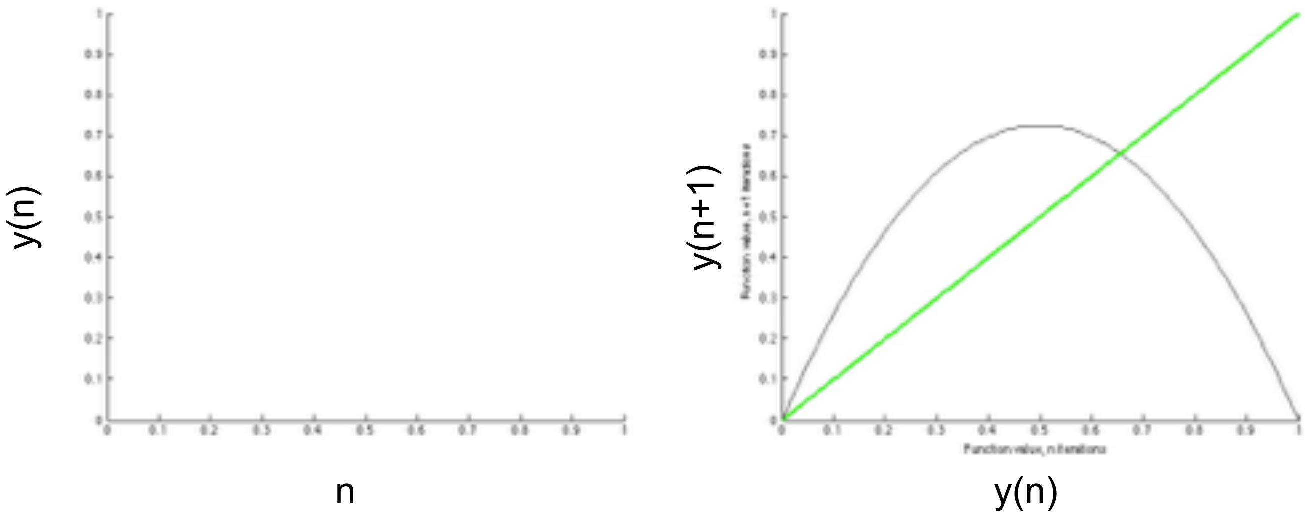
unstable for $r > 3$

R. May., Nature (1976) 261, 459-67

The spectacular consequences of a small bit of non-linearity....

$$y(n+1) = ry(n)(1 - y(n))$$

y(0) = 0.9, r = 2.9



For $x^* = 1 - \frac{1}{r}$, $f'(x^*) = 2 - r$. So . . .

stable for $1 < r < 3$

unstable for $r > 3$

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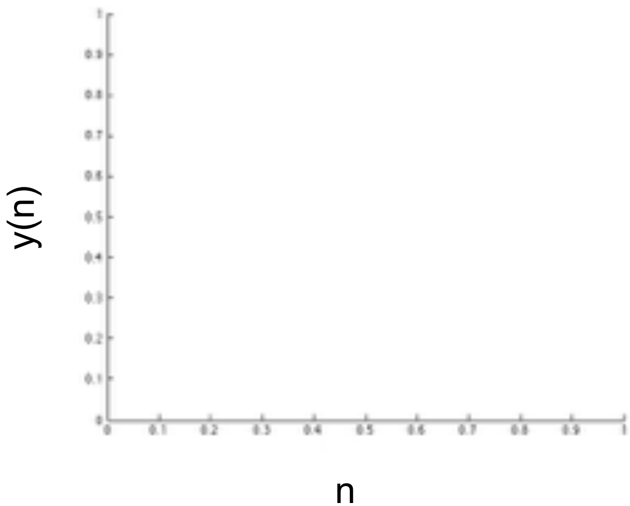
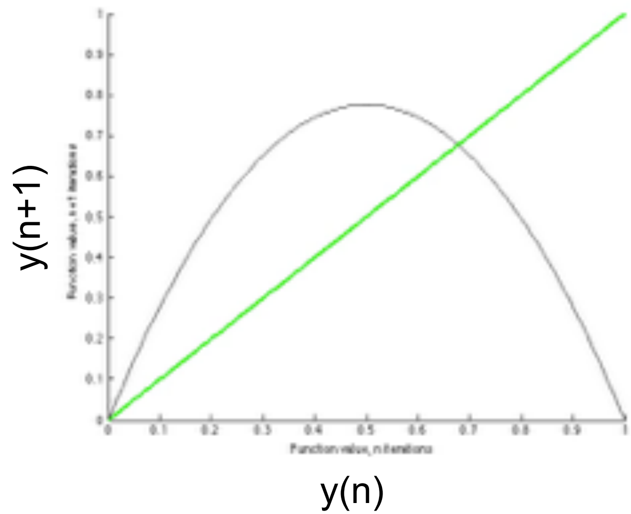
The spectacular consequences of a small bit of non-linearity....

$$y(n+1) = ry(n)(1 - y(n))$$

y(0) = 0.9, r = 3.1





So, this is called a **2-cycle**. That is, both fixed points have lost stability, and we have a system that is said to have **bifurcated**.

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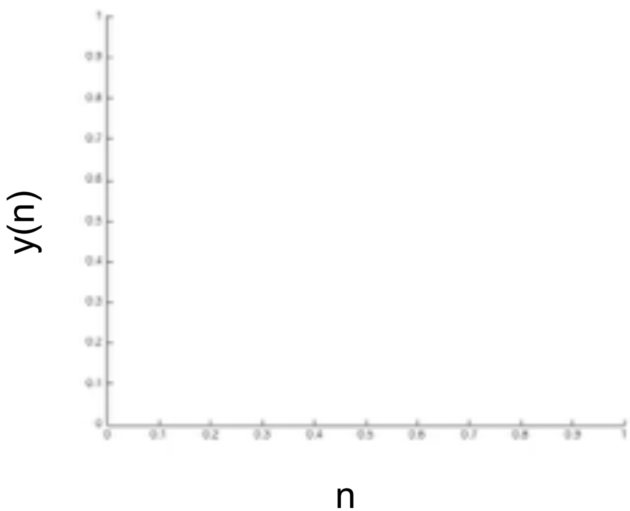
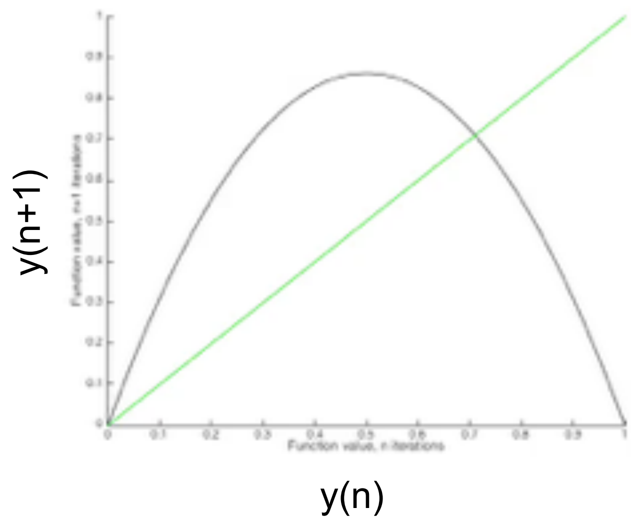
$$y(n+1) = ry(n)(1 - y(n))$$

y(0) = 0.9, r = 3.45





Now…what happened? Even the two-cycle has lost stability!
What do we have now?

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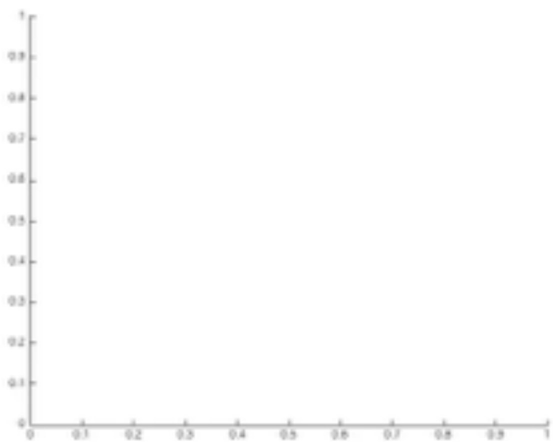
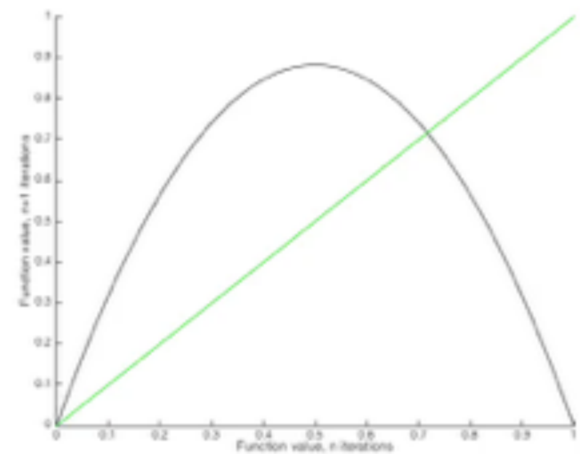
The spectacular consequences of a small bit of non-linearity....

$$y(n+1) = ry(n)(1 - y(n))$$

y(0) = 0.9, r = 3.53





So…a **4-cycle**. The system is said to have bifurcated again, or period doubling

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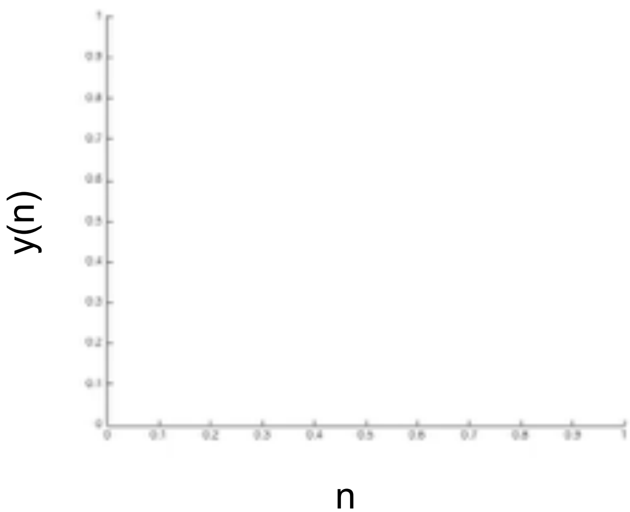
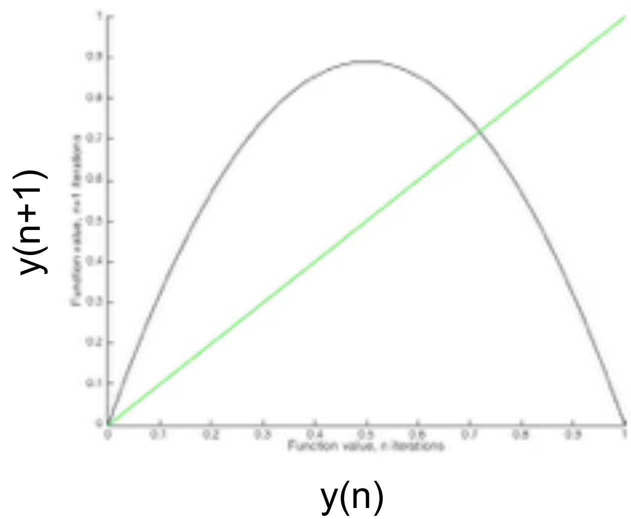
The spectacular consequences of a small bit of non-linearity....

$$y(n+1) = ry(n)(1 - y(n))$$

y(0) = 0.9, r = 3.56





And…a 8-**cycle**. Do you notice that the intervals over which our system bifurcates is getting smaller and smaller?

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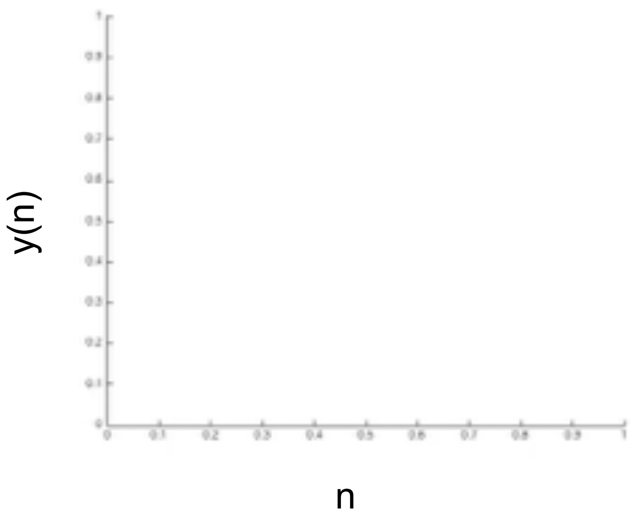
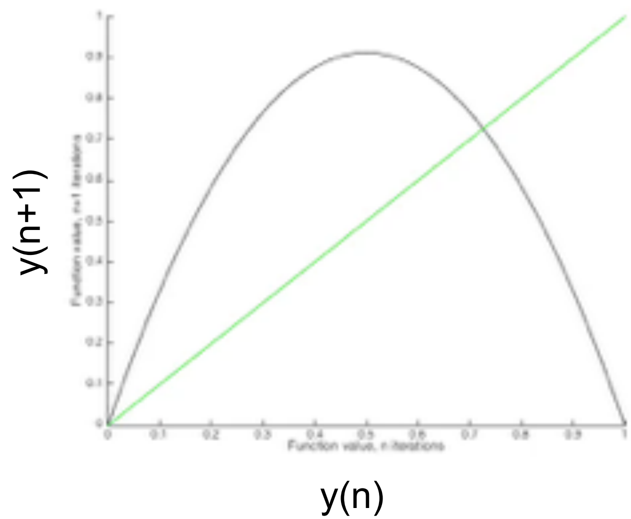
The spectacular consequences of a small bit of non-linearity....

$$y(n+1) = ry(n)(1 - y(n))$$

y(0) = 0.9, r = 3.6





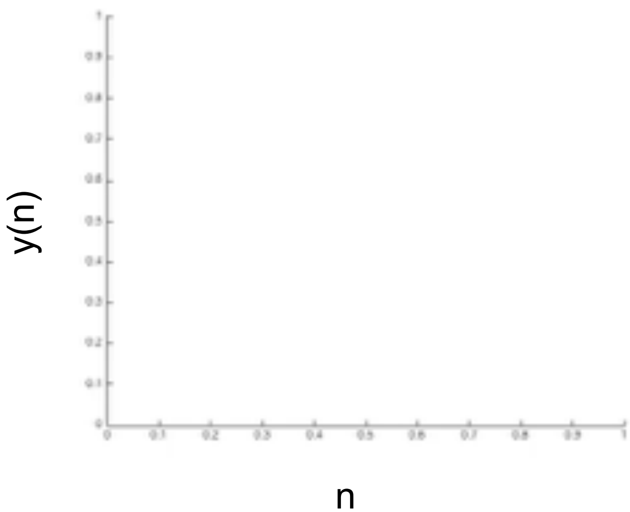
And then we come to this….a regime of so-called **deterministic chaos.**

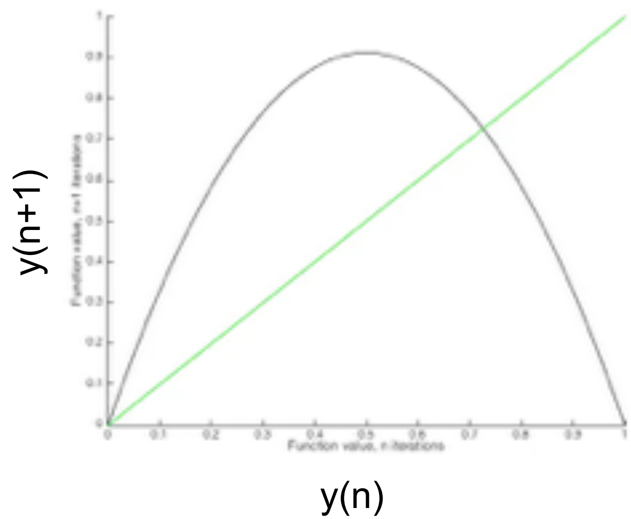
R. May., Nature (1976) 261, 459-67

The spectacular consequences of a small bit of non-linearity....

$$y(n+1) = ry(n)(1 - y(n))$$

y(0) = 0.9, r = 3.6

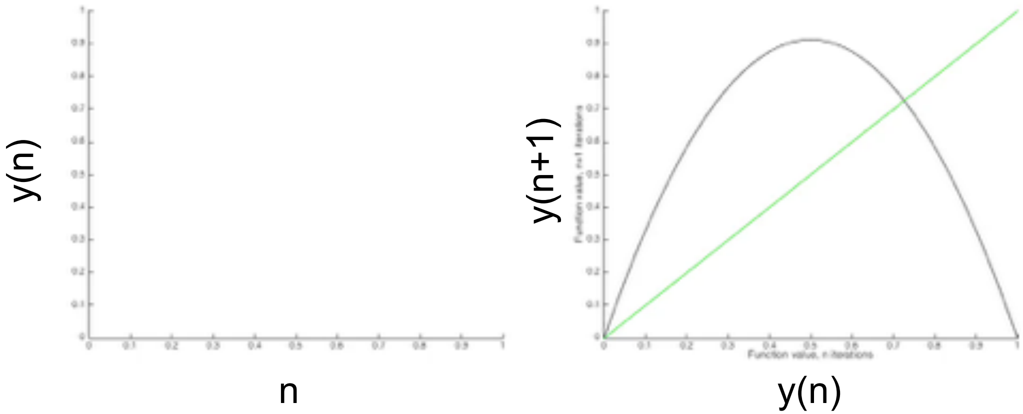




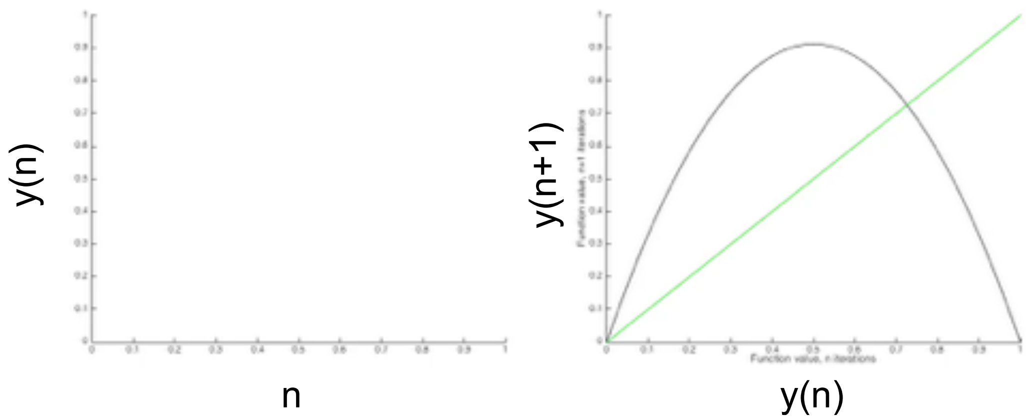
And then we come to this….a regime of so-called **deterministic chaos.** This is characterized by two things: (1) a large number of seemingly constantly changing states, and (2) extreme sensitivity to initial conditions.

R. May., Nature (1976) 261, 459-67

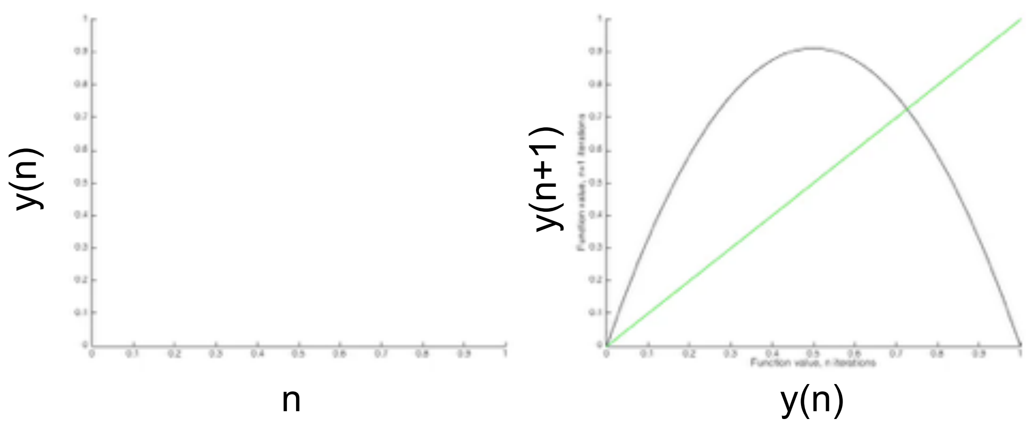
Deterministic chaos: sensitivity to initial conditions…. (the “butterfly effect”)



y(0) = 0.9, r = 3.6



y(0) = 0.8999, r = 3.6

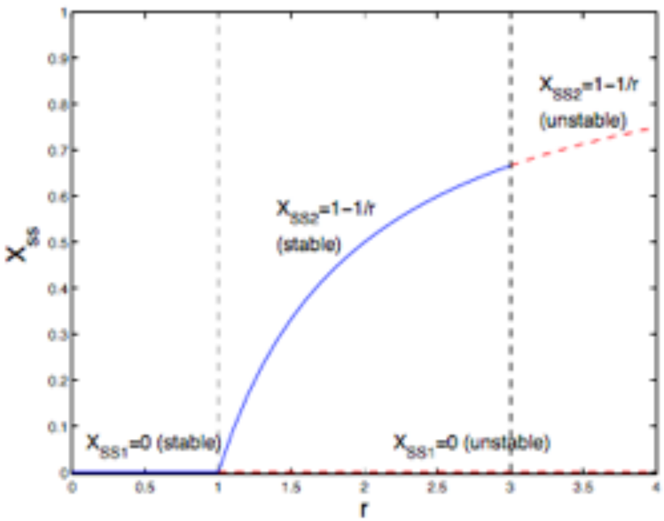


y(0) = 0.9001, r = 3.6

R. May., Nature (1976) 261, 459-67

Deterministic chaos: the number of states….

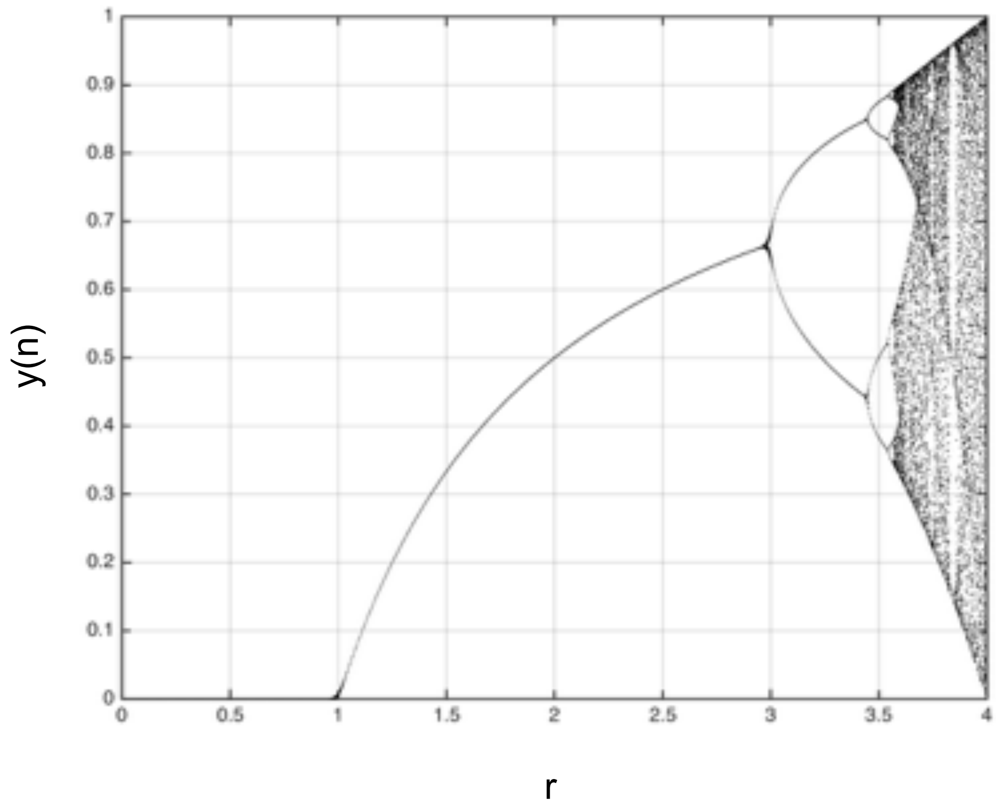
$$y(n+1) = ry(n)(1 - y(n))$$



So…what does happen for **r > 3**?

Deterministic chaos: the number of states….

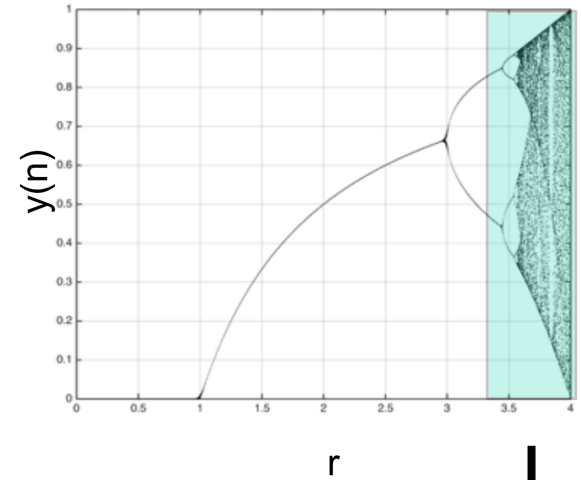
$$y(n+1) = ry(n)(1-y(n))$$

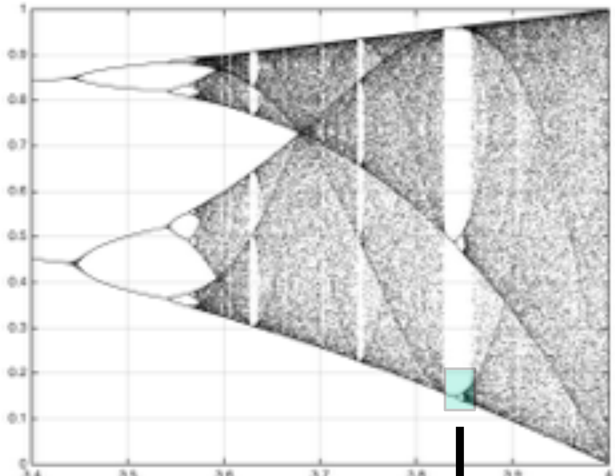


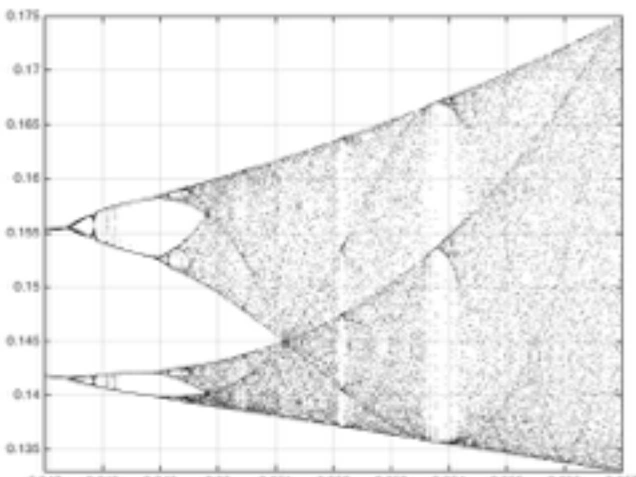
The famous diagram of **period doublings**….

Deterministic chaos: the number of states….

$$y(n+1) = ry(n)(1-y(n))$$



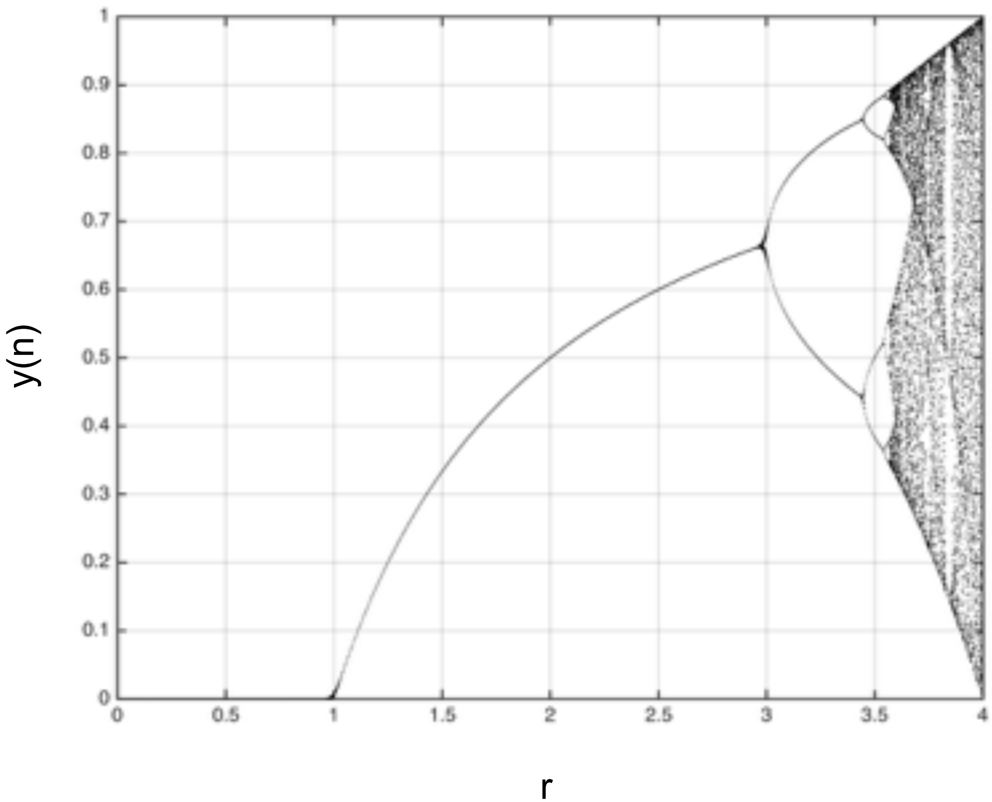




The notions of **self-similarity** and **scale invariance**….

Deterministic chaos: the number of states….

$$y(n+1) = ry(n)(1-y(n))$$



The famous diagram of **period doublings**….can we mathematically understand every doubling point and the entrance into the regime of chaos?

Next, we will further analyze the simple non-linear oscillator systems…

	n = 1	n = 2 or 3	n >> 1	continuum
Linear	exponential growth and decay single step conformational change fluorescence emission pseudo first order kinetics	second order reaction kinetics linear harmonic oscillators simple feedback control sequences of conformational change	electrical circuits molecular dynamics systems of coupled harmonic oscillators equilibrium thermodynamics diffraction, Fourier transforms	Diffusion Wave propagation quantum mechanics viscoelastic systems
Nonlinear	fixed points bifurcations, multi stability irreversible hysteresis overdamped oscillators	anharmomic oscillators relaxation oscillations predator-prey models van der Pol systems Chaotic systems	systems of non-linear oscillators non-equilibrium thermodynamics protein structure/ function neural networks the cell ecosystems	Nonlinear wave propagation Reaction-diffusion in dissipative systems Turbulent/chaotic flows

adapted from S. Strogatz